THE FEBRUARY MEETING OF THE SAN FRANCISCO SECTION.

THE ninth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held at Stanford University, on Saturday, February 24, 1906. The following sixteen members were present:

Professor R. E. Allardice, Mr. A. J. Champreux, Professor R. L. Green, Professor M. W. Haskell, Mr. Edwin Haviland, Professor L. M. Hoskins, Professor D. N. Lehmer, Dr. J. H. McDonald, Dr. W. A. Manning, Professor G. A. Miller, Professor H. C. Moreno, Dr. T. M. Putnam, Mr. Arthur Ranum, Professor Irving Stringham, Professor A. W. Whitney, Professor E. J. Wilczynski.

The attendance also included a number of teachers of mathematics and physics who are not members of the society. A morning and an afternoon session were held, Professor Allardice acting as chairman at both sessions. It was agreed to hold the next meeting at the University of California on September 29, 1906.

The following papers were read at this meeting:

(1) Dr. J. H. MCDONALD: "The theory of the reduction of hyperelliptic integrals of the first kind and of genus 2 to elliptic integrals by a transformation of the nth order."

(2) Dr. W. A. MANNING: "On multiply transitive groups."

(3) Mr. ARTHUR RANUM: "A new kind of congruence group and its application to the group of isomorphisms of any abelian group" (preliminary report).

(4) Professor D. N. LEHMER: "On the orderly listing of substitutions."

(5) Professor D. N. LEHMER: "Note on the values of z of given modulus which give maximum or minimum values to the modulus of a given rational integral function of z."

(6) Professor R. E. ALLARDICE: "Note on Legendre's equation."

(7) Professor R. E. ALLARDICE: "On the multiple points of unicursal curves."

(8) Professor E. J. WILCZYNSKI: "Outline of a projective differential geometry of curved surfaces" (preliminary report).

(9) Mr. E. T. BELL: "Method of dealing with the problems connected with prime numbers" (preliminary report).

(10) Dr. T. M. PUTNAM: "Theorems on perfect numbers."

(11) Dr. J. H. MCDONALD: "A method of simultaneous approximation to two consecutive roots of an algebraic equation of degree n all whose roots are real."

(12) Dr. J. H. MCDONALD: "Remarks on the calculation of roots of Bessel functions."

(13) Professor M. W. HASKELL: "On collineations" (preliminary report).

(14) Professor G. A. MILLER: "Groups in which every subgroup of composite order is invariant."

(15) Professor G. A. MILLER: "The groups which contain exactly thirteen operators of order 2."

Mr. Bell was introduced by the Secretary of the Section. Professor Miller's second paper has appeared in the March number of the BULLETIN. Abstracts of the other papers are given below. The abstracts are numbered to correspond to the titles in the list above.

1. According to the Weierstrass-Picard theorem, when there corresponds to the algebraic relation $y^2 = R_6(x)$ (where $R_6(x)$ denotes a polynomial of the 6th degree) a single reducible integral, then a set of normal integrals may be found, both being reducible, for which the table of canonical periods is known. Dr. McDonald examined the problem in purely algebraic formulation and established a theorem which concerns the relations between the two reducible integrals and shows how to construct either when the other and its reducing substitution are known.

2. If a doubly transitive group of degree n has $r+1$ transitive subgroups H, H_1, $\cdots$, H_r, of degrees q, $q+q_1$, $\cdots$, $q+q_1+\cdots+q_r$, less than n, it is possible, as Dr. Manning proves, to group the letters of H q by q_i in systems of imprimitivity in at least

$$q_i\left(1+\frac{1}{q_r}+\frac{1}{q_{r-1}}+\cdots+\frac{1}{q_i}\right)$$

distinct ways, for $i=1, 2, \cdots, r$. This is a generalization on the theorems of Jordan and Marggraff on primitive groups with transitive subgroups of lower degree. A number of more

special theorems on multiply transitive groups are given in this paper.

3. Mr. Ranum proved that the group of isomorphisms I of any abelian group G of order p^m having n invariants can be expressed as a linear congruence group in n variables, such that each row (or column) of a matrix has its own modulus, namely the corresponding invariant. A general formula is obtained for the order of any such group I. Its series of composition is found; and if $p > 3$, I is shown to be solvable when, and only when, no two of its invariants are equal, while if $p = 2$ or 3, I is solvable when no three of its invariants are equal.

Corresponding to every characteristic subgroup H of G, it is shown that I has an invariant subgroup J whose isomorphisms transform all the operators of H into themselves and whose matrices are congruent to identity when the moduli are the invariants of H. Moreover, corresponding to the characteristic quotient group H' of the same type as H, I has an invariant subgroup J' of the same type as J, whose matrices are obtained from those of J by interchanging rows and columns.

Finally a still more general class of linear congruence groups is mentioned, in which each coefficient of a matrix has its own modulus, and it is shown that if all the moduli in the same rows are equal, the group is necessarily the group of isomorphisms of some abelian group.

4. Professor Lehmer discussed a notation by means of which a substitution is determined by a single number and the remarkable laws arising from the notation in the resulting multiplication tables of various groups were brought out. A method was explained for constructing the multiplication table of the symmetric group on any number of letters with no computation.

5. In Professor Lehmer's second paper certain theorems were brought out concerning the family of curves into which a system of concentric circles are transformed by means of a rational integral function $f(z)$. In particular the locus of the feet of normals from the origin on this family of curves was discussed and the theorem proved that not only the curve passes through the root points of $f(z) = 0$ but that the line joining a root point of $f(z) = 0$ to the origin is tangent to this locus at that point or else forms part of the locus. The paper will be offered to the *Annals of Mathematics* for publication.

6. The equation referred to by Professor Allardice is the linear differential equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0.$$

In Forsyth's Differential Equations (third edition, pages 155 and following), a method is given for finding a relation between the two solutions of this equation, by putting $y = uv - w$, where v stands for one of the solutions. The expression obtained for u is $\frac{1}{2}\log\{(x+1)/(x-1)\}$, and w is obtained in infinite series. This solution is stated to be valid in all cases, except the critical cases where $2n$ is a positive or negative odd integer; though there seems to be no good reason for excluding these cases, if the solution is valid in all other cases. Another form of w is obtained as a sum of functions P_r; but this solution is restricted to the case where n is a positive integer. Now the supposed solution vanishes identically in every case, except when n is a positive integer. In the latter case, certain terms become indeterminate; if these be excluded, a correct solution is obtained, but if they be included, the expression vanishes identically. In the expression in terms of functions P_r, only one term becomes indeterminate, and the limit of this term is Q_n; and thus the expression for Q_n in terms of functions P_r is obtained.

7. The second paper by Professor Allardice was concerned with the calculation of the multiple points of a curve when the coördinates of the curve are given explicitly as rational functions of a parameter. A simple proof was first given of the known theorem that the double point of a cubic is given by the hessian (or canonizant) of a certain binary cubic. By the same method an equation of the sixth degree was then obtained for the three double points of a quartic, in the form of a determinant of the third order. A quartic curve in space was next considered, and the condition for the existence of a double point was shown to be the vanishing of the invariant I_3 of a certain binary quartic. From this condition another form of the equation for the multiple points of a plane quartic was deduced, namely the equation obtained by equating to zero a certain canonizant of two binary quartics. The canonical forms of the quartics obtained by means of this canonizant are $(A, B, C, D, E)(x, y)^4$ and $(A', \lambda B, \lambda C, \lambda D, E')(x, y)^4$.

8. The system of differential equations

$$(1)\quad y_{uu} + 2ay_u + 2by_v + cy = 0,\quad y_{vv} + 2a'y_u + 2b'y_v + c'y = 0,$$

where $y_u = \partial y/\partial u$, $y_v = \partial y/\partial v$, $y_{uu} = \partial^2 y/\partial u^2$, etc., has four linearly independent solutions, which may be taken as homogeneous coördinates of a point on a surface. The curves $u = \text{const.}$ and $v = \text{const.}$ will be asymptotic curves upon this surface. Professor Wilczynski has begun the investigation of the invariants and covariants of (1), which theory, on its geometric side, is equivalent to a projective differential geometry of surfaces.

If the independent variables are not transformed, while y is put equal to $\lambda\bar{y}$, a new system is obtained from (1), satisfied by $\bar{y}$. The invariant formations (semi-invariants and semi-covariants) for this transformation are

$$(2)\quad a',\ b,\ e=a_v-b'_u,\ f=c-a_u-a^2+2bb',\ g=c'-b'_v-b'^2+2aa'$$

and

$$(3)\quad \begin{aligned} &y,\ z = y_u + ay,\quad \rho = y_v + b'y,\\ &\sigma = y_{uv} + b'y_u + ay_v + \tfrac{1}{2}(a_v + b'_u + 2ab')y. \end{aligned}$$

The invariants are functions of the quantities (2) left unchanged by arbitrary transformation of the independent variables. a', b, and e are themselves such invariants. All of them may be expressed in terms of these three and of

$$(4)\quad h=b^2(f-b_v)-\tfrac{1}{4}bb_{uu}+\tfrac{5}{16}b_u^2,\ k=a'^2(g-a'_u)-\tfrac{1}{4}a'a'_{vv}+\tfrac{5}{16}a_v'^2$$

by the help of certain operations involving differentiation processes.

The geometric interpretations are based upon the author's theory of ruled surfaces. In fact, there is associated with every asymptotic curve the ruled surface made up of the asymptotic tangents of the second set along this curve. For example, if $e = 0$, the osculating hyperboloids of the two ruled surfaces, which thus correspond to a given point of the given surface, coincide. If a' or b vanishes, the surface itself is ruled.

9. Mr. Bell's paper showed that the sifting of the primes from one to infinity can be made to depend on certain arith-

metical progressions, whose first terms are the squares of the odd numbers taken in order, and whose common differences are the odd numbers, also taken in order. This method gives all the primes uniformly, and can be used as a starting point from which to determine the number of primes less than a given integer. It is unnecessary to assume the knowledge of any prime number except 2 in dealing with the two problems of determining the primes from one to infinity, and the number of primes between given limits.

10. Dr. Putnam proved in his paper on perfect numbers that a number containing k distinct prime factors can not be perfect if all its primes are equal to or greater than $\frac{3}{2}k+1$, the number being indeed deficient in this case. Odd perfect numbers being necessarily of the form $p^{\alpha}A^2$, it was shown that if the primes in A all occur to the power 2, or all occur to the power four the number cannot be perfect.

11. The method of approximation developed in Dr. McDonald's second paper consists of a chain of quadratic equations depending on three arbitrary numbers, whose roots simultaneously converge to two consecutive roots of an equation of degree n all of whose roots are real.

12. Dr. McDonald's third paper was devoted to the following points:

(1) Genus of the function $J_n(x)$, Rayleigh's determination of least roots. (2) Calculation of early roots. (3) Determination of limit to value of possibly occurring common roots. (4) Relations among the polynomials deduced from recurring relations between Bessel functions.

13. Professor Haskell gave explicit formulas for reducing the collineations leaving a conicoid unchanged to Cremona substitutions in the plane, in such a way as to afford a basis for enumeration of the groups of such collineations, showing that the orders of such groups would be mn, or $2mn$, where m and n may take all values of the orders of groups of linear fractional substitutions of m variables. He developed the following theorem:

Any group leaving a conicoid unchanged is the product or twice the product of two groups of linear fractional substitutions on two independent variables; and conversely, any two

such groups of linear fractional substitutions will generate a group leaving a conicoid unchanged.

14. Professor Miller's paper is devoted to a complete determination of the groups in which every subgroup of composite order is invariant but some subgroups of prime order are non-invariant. If the order of such a group is not a power of a single prime number it is of the form pq^2, where $p > 2$, $q + 1$ is divisible by p, and p, q are primes. The subgroup of order q^2 is of type (1,1) and is the only subgroup of composite order. Moreover, when a non-abelian group contains only one subgroup of composite order the order of the group is pq^2. The necessary and sufficient condition that every subgroup of composite order in a non-abelian and non-hamiltonian group of order p^m, $m > 5$, is invariant is that the group contains invariant operators of order p^{m-2}. If p is odd this condition is necessary and sufficient for every value of $m > 2$, and there are just two such groups for every value of m. When $p = 2$, there is one additional group when $m = 5$ and there is only one possible group when $m = 3$, viz., the octic group. Each of these groups of order p^m contains only one invariant subgroup of order p and has a commutator subgroup of order p. With the single exception of the given group of order 32, the group of cogredient isomorphisms is of order p^2. For the group of order 32 it is of order 16. The paper has been offered to the *Archiv der Mathematik und Physik* for publication.

G. A. MILLER,
Secretary of the Section.

AN APPLICATION OF THE THEORY OF DIFFERENTIAL INVARIANTS TO TRIPLY ORTHOGONAL SYSTEMS OF SURFACES.

BY J. E. WRIGHT, M. A.

(Read before the American Mathematical Society, December 29, 1905.)

IT has been proved by Darboux * that a family of surfaces which makes part of a triply orthogonal system must satisfy a differential equation of the third order. This differential equa-

* "Sur les surfaces orthogonales" (*Bulletin de la Société philomath.*, 1866, p. 16), (*Annales de l' Ecole normale*, 1re séries, vol. 3 (1866), p 97); see Leçons sur les systèmes orthogonaux, pp. 13–14 for complete bibliography.

tion is given by Darboux, in his Leçons sur les systèmes orthogonaux, page 20, in the form $S = 0$, where S is a certain six-rowed determinant. Now, since $S = 0$ has a geometric interpretation, it is natural to expect that S is a differential invariant. If so, it must be expressible as an algebraic invariant of certain forms which may be readily written down.* In this note S is expressed as an algebraic invariant of the forms in question and certain immediate consequences are given.

The determinant S contains only third and lower derivatives of u with respect to x, y, and z, where $u =$ constant is the family of surfaces considered and x, y, z are rectangular cartesian coördinates. Hence the only possible algebraic forms of which it can be an invariant are

$$\sum_{r=1}^{3}\sum_{s=1}^{3} a_{rs}U_rU_s, \text{ and } S_1, S_2, S_3,$$

where the notation of the author's paper already quoted is used. Now with the particular set of variables used, namely rectangular cartesians, the first of these forms becomes $U_1^2 + U_2^2 + U_3^2$, and the others

$$\left(U_1\frac{\partial}{\partial x} + U_2\frac{\partial}{\partial y} + U_3\frac{\partial}{\partial z}\right)^{\lambda} u,$$

when λ takes the values 1, 2 and 3.

It is noticeable that u_{ik} and certain quantities A_{ik} turn up symmetrically in S. Since we have a form $\sum_i \sum_k u_{ik}U_iU_k$, it is suggested that a covariant $\sum_i \sum_k A_{ik}U_iU_k$ is required. Now †

$$A_{ik} = \sum_{l=1}^{3}(u_lu_{ikl} - 2u_{il}u_{kl});$$

we use the ordinary symbolic notation for algebraic invariants and put

$$l_x = S_1, \quad a_x'^2 = b_x'^2 = S_2, \quad a_x^3 = S_3,$$

$$a_x^2 = b_x^2 = \sum_{r=1}^{3}\sum_{s=1}^{3} a_{rs}\,U_r\,U_s.$$

* See a paper by the author, "The differential invariants of space," *Amer. Jour. of Math.*, vol. 27 (1905), pp. 335–336.

† Darboux, Leçons sur les systèmes orthogonaux, p. 19.

Also let

$$h_x^2 = \sum_{i=1}^{3} \sum_{k=1}^{3} A_{ik}\, U_i\, U_k,$$

then it is not difficult to show that

$$h_x^2 = (aab)\,(lab)a_x^2 - 2a_x'b_x'(a'ab)\,(b'ab),$$

provided

$$a_x^2 = U_1^2 + U_2^2 + U_3^2.$$

Hence, since h_x^2 is expressed as a covariant, we have its form for a general a_x^2. The determinant S may now be readily shown to be an invariant of the three quadratics a_x^2, h_x^2 $a_x'^2$ and the one linear form l_x. Its symbolic expression is

$$(hal)\,(ha'l)\,(aa'l).$$

This invariant has a simple geometric interpretation in connection with the ternary forms. Equated to zero it is the condition that the straight line l_x meets the three conics a_x^2, $a_x'^2$, and h_x^2 in six points in involution.

Now that S is expressed as an invariant of the algebraic forms, its expression may be obtained in terms of generalized coordinates ρ_1, ρ_2, ρ_3. The differential invariant theory shows that it is the same algebraic invariant of certain other forms which may be readily obtained. In fact, if $a_x^2 = aU_1^2 + bU_2^2 + cU_3^2 + 2fU_2U_3 + 2gU_3U_1 + 2hU_1U_2$,

$$S_1 = \sum_{i=1}^{3} U_i \frac{\partial u}{\partial \rho_i},$$

$$\Delta S_{m+1} = \begin{vmatrix} \sum_{i=1}^{3} U_i \dfrac{\partial S_m}{\partial \rho_i}, & F_1, & F_2, & F_3 \\ \dfrac{\partial S_m}{\partial U_1}, & a, & h, & g \\ \dfrac{\partial S_m}{\partial U_2}, & h, & b, & f \\ \dfrac{\partial S_m}{\partial U_3}, & g, & f, & c \end{vmatrix}$$

for $m = 1, 2$.

The expression F_i is equal to

$$\sum_{r=1}^{3}\sum_{s=1}^{3}\begin{bmatrix} r & s \\ i & \end{bmatrix} U_r U_s,$$

where

$$\begin{bmatrix} r & s \\ i & \end{bmatrix} = \tfrac{1}{2}\left\{\frac{\partial a_{ri}}{\partial \rho_s} + \frac{\partial a_{si}}{\partial \rho_r} - \frac{\partial a_{rs}}{\partial \rho_i}\right\}$$

is Christoffel's three index symbol, and Δ is the discriminant of a_x^2.

We may use the generalized expression thus obtained to find the condition that the parametric surfaces $\rho_3 =$ constant may form part of a triply orthogonal system. This condition is however too long to be given here.

As another example we may find the condition that the surfaces $u\,(x, y, z) =$ constant are all minimal. This condition, in cartesian coordinates, is

$$(u_{11} + u_{22} + u_{33})(u_1^2 + u_2^2 + u_3^2) = u_{11}u_1^2 + u_{22}u_2^2 + u_{33}u_3^2 + 2u_{23}u_2u_3 + 2u_{31}u_3u_1 + 2u_{12}u_1u_2,$$

where suffixes denote differentiations.

Translated into symbolic notation it becomes $(aa'l)^2 = 0$. Hence the condition expresses that a certain straight line cuts two conics harmonically.

If we take our fundamental quadratic form a_x^2 to be $\sum_{i=1}^{3} H_i^2 U_i^2$ and our minimal system $\rho_3 =$ constant, this invariant condition gives $H_1\,\partial H_2/\partial \rho_3 + H_2\,\partial H_1/\partial \rho_3 = 0$, and hence, of a triply orthogonal system, the parametric surfaces $\rho_3 =$ constant are minimal if H_1H_2 is a function of ρ_1 and ρ_2 only, where the element of length is given by

$$ds^2 = H_1^2 d\rho_1^2 + H_2^2 d\rho_2^2 + H_3^2 d\rho_3^2.$$

BRYN MAWR,
January, 1906.

SURFACES GENERATED BY CONICS CUTTING A TWISTED QUARTIC CURVE AND AN AXIS IN THE PLANE OF THE CONIC.

BY PROFESSOR VIRGIL SNYDER.

(Read before the American Mathematical Society, February 24, 1906.)

THE surface generated by the family of ∞^1 conics which cut five director curves is the simplest generalization of a scroll. Various formulas have been derived, particularly by Stuyvaert,* and the case of a unicursal quintic as directrix was first given by Bertini,† and further developed by Nugteren.‡ It is the purpose of this note to mention an interesting configuration to which the preceding methods do not apply.

1. Given a rational non-singular twisted quartic curve c_4, and a straight line l not cutting it. Let the planes π through l and the points P on l be in $(1, 1)$ correspondence. A plane π will cut c_4 in four points Q_i which with P uniquely determine a conic c_2. It is required to determine the order of the surface F generated by c_2 when π describes the axial pencil about l. Let R_2 be the quadric upon which c_4 lies; let κ, κ' be the two points in which l cuts R_2, and s_3, s_3' the trisecants of c_4 passing through them respectively. The plane (l, s_3) will cut c_4 in a fourth point Q_4 and c_2 will consist of s_3 and $Q_4 P$, if it be assumed that P is not at κ. The second generator of R_2 in this plane is the line $Q_4\kappa'$ and does not therefore lie on F unless P is at κ'. Similarly, the lines s_3' and $Q_4'P'$ will make up a second conic.

Any conic c_2 cuts R_2 in four points, all lying on c_4, hence it can cut no trisecant apart from points on the quartic curve. When P is at κ, the corresponding c_2 has five points on R_2, hence lies entirely on R_2, and also when P is at κ' the conic in its plane lies on R_2. Since every trisecant of c_4 lies on R_2 it follows that every such trisecant will cut five conics of the system, hence *F is of order five.*

Since any plane through l contains only c_2 besides l, it follows that *l is a triple line on F_5.* The complete intersection of F_5

* "Etude de quelques surfaces algébriques engendrées par des courbes du second et du troisième ordre;" dissertation, Gand, 1902.

† "Sulle curve gobbe razionali del quinto ordine," in the Collectanea Mathematica in memoriam D. Chelini, Mediolani, 1881, pp. 313-326.

‡ "Rationale Ruimtekrommen van de fijde Orde;" dissertation, Utrecht, 1902.

and R_2 consists of c_4, $2c_2$, s_3 and s_3', making a configuration of order 10 having fourteen actual double points and two triple points, the latter being κ, κ'.

Other straight lines lie on F_5 also. If in any plane π two points of c_4 be collinear with P, the line joining them and the line joining the other points of c_4 in that plane will constitute the conic. To determine the number of such planes, consider the three planes ω formed by l and a bisecant of c_4 passing through P. To every position of π correspond three planes ω. Conversely, in any plane ω are four points of c_4, making six bisecants cutting l. Each bisecant determines a point P, and thus uniquely fixes π. Between ω, π exists a (6, 3) correspondence, having therefore nine coincidences. In each plane lie two lines belonging to F_5 and cutting c_4 twice, hence: *F_5 contains eighteen bisecants of c_4.*

The bisecants of c_4 which cut l define a scroll R_9 of order 9, on which l and c_4 are each triple, and s_3, s_3' are triple generators. The intersection of F_5 and R_9 is made up of l counted nine times, c_4 counted three times, the two trisecants which cut l each counted three times, and the eighteen bisecants mentioned above. This may be expressed thus:

$$(F_5,\ R_9) = l(9) + c_4(12) + 2s_3(6) + 18s_2(18).$$

2. In case κ and the plane $(l,\ s_3)$ are corresponding elements, then the residual point in which the (degraded) c_2 cuts l is indeterminate, hence this plane is a factor of F_5. The line l is double on the other factor F_4. We now have

$$(F_4,\ R_2) = c_4(4) + 2s_3(2) + c_2(2),$$

$$(F_4,\ R_9) = c_4(12) + l(6) + 2s_3(6) + 12s_2(12).$$

There are now but six coincidences in the correspondence between ω and π, apart from s_3 which counts for three.

3. If κ, $(l,\ s_3)$ and κ', $(l,\ s_3')$ are both pairs of corresponding elements, the surface is a cubic, on which l is a simple line. The equations are

$$(F_3,\ R_2) = c_4(4) + 2s_3(2),$$

$$(F_3,\ R_9) = c_4(12) + l(3) + 2s_3(6) + 6s_2(6).$$

Of the ten lines on F_3 which cut l, eight lie on R_9; the other two are the residuals in the planes $(l,\ s_3)$, $(l,\ s_3')$ and have but

one point on c_4. F_3 contains sixteen other lines which do not lie on R_2 nor on R_9.

4. Now suppose no restriction be put upon the relation between P and π, but that c_4 passes through κ. The conic in every plane π will pass through κ. The surface is now of order four, and l is a double line upon it. The only conic common to F_4 and R_2 is in the plane π when P is at κ'; that in the plane corresponding to κ touches l at κ, but does not lie on R_2. In the plane of (l, s_3'), l is itself part of c_2. The point κ is a triple point on the surface. The R_9 of bisecants breaks up into a cubic cone K_3 having its vertex at κ and having s_3 for a double edge, and an R_6 having l for triple line, c_4 for double curve, s_3' for a triple generator, and s_3 for a simple generator.*

We now have

$$(F_4, R_6) = c_4(8) + s_3(1) + s_3'(3) + l(6) + 6s_2(6).$$

The correspondence (ω, π) is now (3, 3) with six coincidences, which account for twelve lines on F_4, but only six belong to R_6, the other six lying on K_3,

$$(F_4, K_3) = c_4(4) + s_3(2) + 6s_1(6).$$

The surfaces F_4, K_3 furnish a monoidal representation of c_4 when the common vertex is a point on the curve.

If in the correspondence (P, π), κ, (l, s_3) are corresponding elements, nothing new will result, since the line joining Q_4 of c_4 to κ is not a generator of R_2. The surface is not changed except that one of the six bisecants of c_4 mentioned above now passes through the triple point κ.

5. If κ', (l, s_3') are corresponding elements however, the surface reduces to a cubic on which l is a simple line and κ is a double point;

$$(F_3, R_2) = c_4(4) + 2s_3(2); \quad (F_3, R_6) = l(3) + c_4(8) + s_3(1)$$
$$+ s_3'(3) + 3s_2(3), \quad (F_3, K_3) = c_4(4) + s_3(2) + 3s_1(3).$$

F_3 and K_3 furnish a monoidal representation of c_4.

6. If l is a bisecant of c_4, every c_2 must pass through two fixed points. Since it must also pass through P on l, the latter is a factor of every c_2, and the residual is a straight line joining

* This surface is type 80 in my enumeration of sextic scrolls, *Amer. Jour. of Math.*, vol. 27, p. 101.

the other two points of c_4 in π. The surface is now of order three and l is a double line above it. Since from every point of l two bisecants to c_4 can be drawn, apart from l itself, the surface is a cubic scroll of the first kind, R_3. The R_6 breaks up into a cubic cone, with vertex at κ', and this same R_3.

Finally, if l has three points upon c_4, F becomes the R_2 upon which c_4 lies.

7. If c_4 is of genus one it has no trisecants. As no line can cut more than two conics apart from the two lying on R_2, the surface is of order four and l is a double line upon it. The scroll of bisecants is now R_8 on which l is a double directrix and c_4 a triple curve. The correspondence (π, ω) is now (2, 6) with eight coincidences and 16 lines, hence

$$(F_4, R_8) = c_4(12) + l(4) + 16s_2(16).$$

If one or both points κ, κ' correspond to the plane containing a bisecant passing through them, the conics in these planes break up, but no important changes in the form of the surfaces occur.

8. If l intersects c_4 at κ, R_8 breaks up into an elliptic K_3 and an R_5 having the symbol $l_2^2 + c_4^2$.* F is a cubic on which l is a simple line and κ is a node.

$$(F_3, R_5) = l(2) + c_4(8) + 5s_2(5), \quad (F_3, K_3) = c_4(4) + 5s_1(5).$$

The correspondence (π, ω) is now (2, 3) and the residual lines in the planes of the coincidences belong to K_3. If l intersects c_4 twice the surface reduces to R_2 containing c_4 and l.

9. If c_4 has a node, and no restrictions as to (π, P), the surface is F_4, having l for double line. Two lines through the node cut l and cut c_4 again;

$$(R_8, F_4) = l(4) + c_4(12) + 16s_2(16).$$

If κ (l node) are corresponding elements, the surface is F_3,

$$(F_3, R_8) = l(2) + c_4(12) + 10s_2(10),$$

since the line joining the node to κ and cutting c_4 again counts for three coincidences. Both points κ, κ' in which l cuts K_2 on which c_4 lies cannot give rise to coincidence, because they lie in the same plane κ.

* This scroll is type B, iv of Schwarz's classification in *Crelle's Journal*, vol. 67, p. 37.

10. If l cuts c_4 in κ, the surface is a nodal cubic, having one node at κ, and another at the node of c_4. R_8 breaks up into K_3 and R_5, the latter having the symbol $l_2 + c_4^2 + 3^2$ (Schwarz's A, vii), the double generation being the line joining the node to the fourth point in the plane containing l. Thus,

$$(F_3, R_5) = l(2) + c_4(8) + g_2(2) + 3s_2(3).$$

If c_4 has a cusp, the second nodal point becomes uniplanar. Further specializations result in quadrics and quadric cones.

CORNELL UNIVERSITY,
January, 1906.

OPERATION GROUPS OF ORDER $p_1^{m_1\mu_1}p_2^{m_2\mu_2}$.

BY PROFESSOR O. E. GLENN.

IT is desired to make certain generalizations concerning the groups of order the product of powers of two primes p_1, p_2, such that $p_1 \equiv 1 \pmod{p_2}$, these groups possessing abelian subgroups H_i of type $[\mu_i, \mu_i, \cdots, \mu_i]$ $(i = 1, 2)$. It is possible to specify for these groups those subgroups (here called basic subgroups) from which it is necessary and sufficient that generating operations be selected in order that they may generate the whole group G. This general problem connected with groups of composite order seems to merit more attention than it has thus far received. If

$$H_1 = \{P_1, P_2, \cdots, P_{m_1}\}, \quad H_2 = \{Q_1, Q_2, \cdots, Q_{m_2}\},$$

then the number of operations of order $p_i^{\mu_i}$ in H_i is

$$\sum_{j=0}^{m_i-1} {}_{m_i}C_j [p_i^{\mu_i} - \Phi(p_i^{\mu_i})]^j [\Phi(p_i^{\mu_i})]^{m_i-j}$$

$$= [p_i^{\mu_i-1} + \Phi(p_i^{\mu_i})]^{m_i} - p^{m_i(\mu_i-1)},$$

so that the number of cyclical subgroups of order $p_i^{\mu_i}$ in H_i is

$$N_{p_i}^{\mu_i} = \frac{p_i^{m_i(\mu_i-1)}(p_i^{m_i} - 1)}{\Phi(p_i^{\mu})}$$

$$= p_i^{(m_i-1)(\mu_i-1)}(p_i^{m_i-1} + p_i^{m_i-2} + \cdots + p_i + 1).$$

Let us now suppose that $\mu_1 = \mu_2 = 1$. Transformation of the $N^1_{p_1}$ subgroups of H_1 by Q_κ permutes them in sets of p_2 or else leaves them invariant. If ρ_{Q_κ, p_1} is the number left invariant

$$N^1_{p_1} - \rho_{Q_\kappa, p_1} \equiv 0 \,(\mathrm{mod}\, p_2),$$

and when $p_1 \equiv 1 \,(\mathrm{mod}\, p_2)$, $\rho_{Q_\kappa, p_1} \geqq m_1$. It follows that P_λ $(\lambda = 1, 2, \cdots, m_1)$ can be so selected that

$$Q_1^{-1} P_\lambda Q_1 = P_\lambda^{\alpha^{x_{1\lambda}}}, \quad \alpha^{p_2} \equiv 1 \,(\mathrm{mod}\, p_1). \tag{1}$$

Now H_1 is self-conjugate * in G hence

$$Q_2^{-1} P_\lambda Q_2 = P_1^{a_{1\lambda}} P_2^{a_{2\lambda}} \cdots P_{m_1}^{a_{m_1\lambda}},$$

and

$$(Q_1 Q_2)^{-1} P_\lambda (Q_1 Q_2) = \prod_{\nu=1}^{m_1} P_\nu^{a_{\nu\lambda}\alpha^{x_{1\lambda}}} = (Q_2 Q_1)^{-1} P_\lambda (Q_2 Q_1)$$
$$= \prod_{\nu=1}^{m_1} P_\nu^{a_{\nu\lambda}\alpha^{x_{1\nu}}}.$$

$$a_{\nu\lambda}(\alpha^{x_{1\lambda}} - \alpha^{x_{1\nu}}) \equiv 0 \,(\mathrm{mod}\, p_1), \tag{2}$$

and if $x_{1\lambda} \not\equiv x_{1\nu} \,(\mathrm{mod}\ p_2)$ then $a_{\nu\lambda} \equiv 0 \,(\mathrm{mod}\, p_1)$. Suppose on the other hand that some of the $x_{1\lambda}$ are congruent to some of the $x_{1\nu} (\nu \not\equiv \lambda)$, say $x_{1\lambda} \equiv x_{1\nu} (\nu = \lambda + 1, \lambda + 2, \cdots, \lambda + r)$, then $a_{\lambda+i}$, may be zero or not. Assuming the latter, we prove that *all the subgroups in the group* $\{P_\lambda, P_{\lambda+1}, P_{\lambda+2}, \cdots, P_{\lambda+r}\}$ *are permutable with* Q_1, *and the operations* $P_\lambda, P_{\lambda+1}, P_{\lambda+2}, \cdots, P_{\lambda+r}, Q_2$ *form a subgroup of order* $p_1^{r+1} p_2$.

The first statement is obvious since

$$Q_1^{-1} \prod_{i=0}^{r} P_{\lambda+i}^{a_i} Q_1 = \left(\prod_{i=0}^{r} P_{\lambda+i}^{a_i} \right)^{\alpha^{x_{1\lambda}}}.$$

To prove the latter we have

$$Q_2^{-1} P_\lambda Q_2 = P_\lambda^{a_{\lambda\lambda}} P_{\lambda+1}^{a_{\lambda+1,\lambda}} \cdots P_{\lambda+r}^{a_{\lambda+r,\lambda}} = \prod_{i=0}^{r} P_{\lambda+i}^{a_{\lambda+i,\lambda}}.$$

Also

$$(Q_1 Q_2)^{-1} P_{\lambda+i} (Q_1 Q_2) = P_1^{a_{1,\lambda+i}\alpha^{x_{1\lambda}}} P_2^{a_{2,\lambda+i}\alpha^{x_{1\lambda}}} \cdots P_{m_1}^{a_{m_1\lambda+i}\alpha^{x_{1\lambda}}}$$
$$= \prod_{j=1}^{m_1} P_j^{a_{j,\lambda+i}\alpha^{x_{1\lambda}}} = (Q_2 Q_1)^{-1} P_{\lambda+i} (Q_2 Q_1)$$
$$= P_1^{a_{1,\lambda+i}\alpha^{x_{11}}} P_2^{a_{2,\lambda+i}\alpha^{x_{12}}} \cdots P_\lambda^{a_{\lambda,\lambda+i}\alpha^{x_{1\lambda}}} \cdots P_{\lambda+r}^{a_{\lambda+r,\lambda+i}\alpha^{x_{1\lambda}}} \cdots P_{m_1}^{a_{m_1\lambda+i}\alpha^{x_{1m_1}}},$$

* Burnside, Finite Groups, p. 351, Theorem V.

and since $$x_{1k} \not\equiv x_{1\lambda} \quad (k \neq \lambda, \lambda+1, \cdots, \lambda+r),$$

$$a_{j,\lambda+i} \equiv 0 \pmod{p_1} \quad (j \neq \lambda, \lambda+1, \cdots, \lambda+r),$$

so that

$$Q_2^{-1} P_{\lambda+i} Q_2 = P_\lambda^{a_{\lambda,\lambda+i}} P_{\lambda+1}^{a_{\lambda+1,\lambda+i}} \cdots P_{\lambda+r}^{a_{\lambda+r,\lambda+i}} = \prod_{j=\lambda}^{\lambda+r} P_j^{a_{j,\lambda+i}} \quad (i = 0, 1, 2, \cdots, r),$$

and the operations P_j, Q_2 form a subgroup (I) of order $p_1^{r+1} p_2$. The number of subgroups of order p_1 in (I) is

$$p_1^r + p_1^{r-1} + \cdots + p_1 + 1,$$

and at least $r+1$ of them are permutable with Q_2, since $p_1 \equiv 1 \pmod{p_2}$. If these are $\{P_j'\}$ $(j = \lambda, \lambda+1, \cdots, \lambda+r)$, then, on replacing P_j by P_j', the series of m_1 independent generators of H_1

$$P_1, P_2, P_3, \cdots, P_{\lambda-1}, P_\lambda', P_{\lambda+1}', \cdots, P_{\lambda+r}', P_{\lambda+r+1}, \cdots, P_{m_1}$$

has the property that each generator is transformed into one of its own powers by both Q_1 and Q_2. Hence, referring back to equation (2), even when $x_{1\lambda} \equiv x_{1\nu} \pmod{p_2}$ we may write $a_{\nu\lambda} \equiv 0 \pmod{p_1}$ $(\nu \neq \lambda)$. It is also obvious that when $\nu = \lambda$, then $a_{\lambda\lambda} \equiv$ some power of α, as $\alpha^{x_{2\lambda}}$. Hence G is defined by the relations

$$P_\lambda^{p_1} = Q_\kappa^{p_2} = 1, \; P_\lambda P_\nu = P_\nu P_\lambda, \; Q_\kappa Q_l = Q_l Q_\kappa, \; Q_\kappa^{-1} P_\lambda Q_\kappa = P_\lambda^{\alpha^{x_{\kappa\lambda}}}$$

$$(\lambda, \nu = 1, 2, \cdots, m_1; \; \kappa, l = 1, 2, \cdots, m_2)$$

$$\alpha^{p_2} \equiv 1 \pmod{p_1}, \quad x_{\kappa\lambda} \equiv 0, 1, \cdots, p_2 - 1 \pmod{p_2}.$$

LEMMA: *There are $m_2 - 1$ of the $x_{\kappa\lambda}$ equal to zero, and no generality is lost by assuming that*

$$x_{11} \equiv x_{21} \equiv x_{31} \equiv \cdots x_{m_2-1,1} \equiv 0, \; x_{m_2 1} \equiv 1 \pmod{p_2}.$$

Since G is by assumption non-divisible, there is at least one operation, as Q_{m_2}, which transforms P_1 into $P_1^a (a \neq 1)$. Say $Q_{m_2}^{-1} P_1 Q_{m_2} = P_1^a$. If $Q_{m_2-\tau}'$ is any other generator of H_2, by what has been given

$$Q_{m_2-\tau}'^{-1} P_1 Q_{m_2-\tau}' = P_1^{a^{x'_{m_2-\tau,1}}} = Q_{m_2}^{-x'_{m_2-\tau,1}} P_1 Q_{m_2}^{x'_{m_2-\tau,1}},$$

$$\therefore \left(Q'_{m_2-\tau} Q^{-x'_{m_2-\tau,1}}_{m_2}\right)^{-1} P_1 \left(Q'_{m_2-\tau} Q^{-x'_{m_2-\tau,1}}_{m_2}\right) = P_1,$$

$$(= Q^{-1}_{m_2-\tau} P_1 Q_{m_2-\tau} \text{ say}).$$

It follows that $Q_{m_2-\tau}$ $(\tau = 1, 2, \cdots, m_2 - 1)$ is permutable with P_1, so that $x_{11} \equiv x_{21} \cdots \equiv x_{m_2-1,1} \equiv 0$ and $x_{m_2 1} \equiv 1$. By this means all the generators of order p_2 of the group G are fixed and no other invariants $x_{\kappa\lambda}$ can be placed equal to zero without destroying the generality of the defining relations.

The most general form of the generating operations of order p_1 of G will now be derived.

Let

$$\bar{P}_t = P_1^{\alpha_{1t}} P_2^{\alpha_{2t}} \cdots P_{m_1}^{\alpha_{m_1 t}}; \quad \bar{Q}_u = Q_1^{\beta_{1u}} Q_2^{\beta_{2u}} \cdots Q_{m_2}^{\beta_{m_2 u}},$$

and write

$$\bar{Q}_u^{-1} \bar{P}_t \bar{Q}_u = \bar{P}_t^{\alpha^{x_{ut}}}.$$

Under the assumption $\alpha_{it} \not\equiv 0 \pmod{p_1}$ we get

$$(3)\quad \beta_{m_2 u} x_{m_2 i} + \beta_{m_2-1,u} x_{m_2-1,i} + \beta_{m_2-2,u} x_{m_2-2,i} + \cdots + \beta_{1u} x_{1i} \equiv x_{ut} \pmod{p_2} \quad (i = 1, 2, \cdots, m_1),$$

so that for a chosen t (say $t = 1$) and a chosen u (as $u = 1$) we have a set of m_1 linear congruences in the β_{ku} (as the following):

$$(4)\quad \left.\begin{aligned} &\beta_{m_2 1} \equiv 0, \\ &\beta_{m_2-1,1} x_{m_2-1,2} + \cdots + \beta_{11} x_{12} \equiv 0, \\ &\beta_{m_2-1,1} x_{m_2-1,3} + \cdots + \beta_{11} x_{13} \equiv 0, \\ &\quad \cdots\cdots\cdots\cdots \\ &\beta_{m_2-1,1} x_{m_2-1,m_1} + \cdots + \beta_{11} x_{1m_1} \equiv 0, \end{aligned}\right\} \pmod{p_2}.$$

If no $\alpha_{i1} \equiv 0$, then assuming $m_1 = m_2$,

$$\begin{vmatrix} x_{m_2-1,2} & \cdots & x_{12} \\ x_{m_2-1,3} & \cdots & x_{13} \\ \cdots & \cdots & \cdots \\ x_{m_2-1,m_1} & \cdots & x_{1m_1} \end{vmatrix} \equiv 0 \pmod{p_2},$$

a relation among the invariants of the group which is not possible *in general*. Hence since we seek *general* generators of the group we must have at least one α_{i1} ($\not\equiv \alpha_{11}$ since $\overline{P}_1$ obviously $= P_1^{\alpha_{11}}$ as a special case) and one of the β_{k1}; viz., $\beta_{m_2 1}$ equal to zero. Again if $m_1 = m_2 + \sigma$ ($\sigma =$ integer), then at least $\sigma + 1$ of the α_{i1} ($\not\equiv \alpha_{11}$) are zero. If $m_1 < m_2$, no α_{i1} is necessarily zero, by virtue of (4). In the same way if $u = 2$, $t = 1$, $\beta_{m_2, 2} \equiv 0$, and in general if $u = u(< m_2) t = 1$, $\beta_{m_2 u} \equiv 0$; the same condition holding for $m_1 \gtreqless m_2$ as above. When $u = m_2$, $t = 1$ we have

$$\beta_{m_2 m_2} x_{m_2 1} \equiv 1, \text{ (whence } \beta_{m_2 m_2} \equiv 1 \pmod{p_2}).$$

$$\left.\begin{aligned} x_{m_2 2} + \beta_{m_2-1 m_2} x_{m_2-1 2} + \cdots + \beta_{1 m_2} x_{12} &\equiv 1, \\ x_{m_2 3} + \beta_{m_2-1 m_2} x_{m_2-1 3} + \cdots + \beta_{1 m_2} x_{13} &\equiv 1, \\ \cdots\cdots\cdots\

$$N = \frac{1}{m_1}\left[\sum_{\sigma=0}^{(m_1-1)(p_2-2)} P(0, 1, \cdots, p_2-2)^{m_1-1}\sigma + \psi\right],$$

where $P(0, 1, \cdots, p_2-2)^{m_1-1}\sigma$ stands for the number of partitions of σ in (m_1-1)'s by the numbers $0, 1, \cdots, p_2-2$; and ψ is a determinate function of p_2 and m_1.

SPRINGFIELD, MO.,
December, 1905.

A DEFINITION OF QUATERNIONS BY INDEPENDENT POSTULATES.*

BY MISS R. L. CARSTENS.

(Read before the American Mathematical Society, February 24, 1906.)

§ 1. *Quaternions with respect to a Domain D.*†

THE usual theory relates to quaternions $a_1 + a_2 i + a_3 j + a_4 k$ in which the coefficients a_i range independently over all real numbers or else over all complex numbers, and the units have the following multiplication table:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

These conditions give the real quaternion system and the octonion system.‡ As an obvious generalization, the coefficients may range independently over all the elements of any domain D.

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*See Dickson, "On hypercomplex number systems," *Transactions Amer. Math. Society*, vol. 6 (1905).

†A domain consists of any class of elements.

‡Octonions may be considered as quaternions with complex coefficients.

§ 2. *The Postulates.*

A set of four ordered elements $a = [a_1, a_2, a_3, a_4]$ of D will be called a quaternion a. The symbol $a = [a_1, a_2, a_3, a_4]$ employed is purely positional, without functional connotation. Its definition implies that $a = b$ if and only if $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$, $a_4 = b_4$.

Postulate I. If a and b are any two quaternions, then $s = [a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4]$ is a quaternion.

Definition. Addition of quaternions is defined by $a \oplus b = s$.

Postulate II. $0 = [0, 0, 0, 0]$ is a quaternion.

Postulate III. If 0 is a quaternion, then to any quaternion a corresponds a quaternion a' such that $a \oplus a' = 0$.

Theorem 1. Quaternions form a commutative group under addition.

Postulate IV. a and b being any two quaternions, then $a \oplus b = p = [p_1, p_2, p_3, p_4]$ is a quaternion, where

$$p_1 = a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4, \quad p_3 = a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2$$

$$p_2 = a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3, \quad p_4 = a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1,$$

if the p_i's are in D.

Definition. The product of two quaternions is defined by $a \otimes b = p$.

Theorem 2. Multiplication is not commutative.

Theorem 3. Multiplication is distributive (right and left).

Theorem 4. Multiplication is associative.

To make quaternions four dimensional we add a fifth postulate:

Postulate V. If $\tau_1, \tau_2, \tau_3, \tau_4$ are elements of D such that $\tau_1a_1 + \tau_2a_2 + \tau_3a_3 + \tau_4a_4 = 0$ for every quaternion a, then $\tau_1 = 0$, $\tau_2 = 0$, $\tau_3 = 0$, $\tau_4 = 0$.

Theorem 5. There exist four quaternions $\epsilon_i = [a_{i1}, a_{i2}, a_{i3}, a_{i4}]$ such that $|a_{ij}| \neq 0$.

§ 3. *Identificction with Ordinary Quaternions.*

The quaternion system as thus defined is holoedrically isomorphic with the quaternions of Hamilton, the coefficients belonging to the same domain D.

The quaternions $e_1 = [1, 0, 0, 0]$, $e_2 = [0, 1, 0, 0]$, $e_3 = [0, 0, 1, 0]$, $e_4 = [0, 0, 0, 1]$ form a four dimensional system since

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \neq 0.$$

By postulate IV and the definition of multiplication, these quaternions e have the multiplication table

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	$-e_1$	e_4	$-e_3$
e_3	e_3	$-e_4$	$-e_1$	e_2
e_4	e_4	e_3	$-e_2$	$-e_1$

which, apart from symbolism, is the same as the table of § 1.

§ 4. *On the Independence of the Postulates.*

If D is a domain admitting addition and subtraction, postulates II and III are redundant.

Aside from this case, postulates I–V are independent as shown by the following systems:

(I) Elements 0, $[\pm 1, 0, 0, 0]$, $[0, \pm 1, 0, 0]$, $[0, 0, \pm 1, 0]$, $[0, 0, 0, \pm 1]$.

(II) D is the domain of positive integers.

(III) Set (II) with 0 added.

(IV) D is the domain of complex numbers, the a_i being pure imaginaries.

(V) a_1 arbitrary; other a's $= 0$.

THE UNIVERSITY OF COLORADO,
BOULDER, COLO.,
February, 1906.

NOTE ON THE HEINE-BOREL THEOREM.

BY MR. N. J. LENNES.

(Read before the Chicago Section of the American Mathematical Society, December 29, 1905.)

In this note is presented a generalized form of the Heine-Borel theorem together with a corollary which is of immediate application in many theorems in metrical analysis. The present form of the theorem is the result of an effort to understand the meaning of the properties of boundedness and unboundedness of sets of points (numbers) in non-metrical analysis situs. It appears that for many purposes the property of boundedness when applied to a closed set may be replaced by the property that the set shall contain its limit points at infinity, *i. e.*, the set shall be closed even if it is unbounded. In non-metrical analysis, therefore, the chief distinction between a closed bounded set and an unbounded set not containing its limit points at infinity seems to be that the latter is necessarily not closed.

The theorem is stated for the case of three dimensions and the language of geometry is used exclusively. In view of the one-to-one correspondence of the set of all points in a three-space and the set of all triples of real numbers, the reader may, if he wishes, regard the geometric language as a notation for a three dimensional number manifold.

§ 1. *Definitions and Preliminary Notions.*

The word region is used to denote any set of points whatever. Two half-lines proceeding from the same point O and not lying in the same line form an angle. (The half-line contains the point O from which it proceeds.) We assume that an angle separates the remaining points of the plane in which it lies into two unique sets, an interior and an exterior set. If the half-lines a, b, c, no two of which lie in the same line and all three of which do not lie in the same plane, proceed from the same point O then the three angles formed by these half-lines together with the interior points of these angles form a trihedron $Oabc$. If A, B, C, are points of the respective half-lines a, b, c, then the four triangles OAB, OBC, OCA, and ABC, together with their interior points, form a tetrahedron. Such tetrahedron we shall speak of as associated with the tri-

hedron *Oabc*. We assume that a trihedron and also a tetrahedron separates space into two unique sets, and hence that these figures have a definite interior.

A finite point P' is a limit point of a set $[P]$ of points if there are points of $[P]$ other than P' within every tetrahedron of which P' is an interior point. If a set $[P]$ of a half-line is unbounded then the set $[P]$ is said to have a limit point ∞ on the half-line. We will regard every half-line as having one point at infinity. A point ∞ on a half-line l proceeding from a point O is said to be a limit point of a set $[P]$ if for every trihedron *Oabc* containing the points of l as interior points, and for every associated tetrahedron $OABC$ there are points of $[P]$ other than ∞ within *Oabc* and exterior to $OABC$. A point is within a region R if it is not a limit point of points not of the region. This definition applies to points at infinity, *i. e.*, a point ∞ lies within a region R if there exists a trihedron *Oabc* and an associated tetrahedron $OABC$ such that ∞ is an interior point of *Oabc* and further such that every point, other than ∞, within *Oabc* and exterior to $OABC$ is a point of R. A closed set contains all its limit points. This also applies to points at infinity.

§ 2. *The Generalized Heine-Borel Theorem.*

THEOREM: *If* $[P]$ *is a closed set of points and if* $[R]$ *is a set of regions such that every point of* $[P]$ *is an interior point of at least one region of the set* $[R]$ *then there exists a finite subset* $R_1, R_2, \cdots, R_n$, *of regions of the set* $[R]$ *such that every point of* $[P]$ *lies within at least one region of the set* $R_1, R_2, \cdots, R_n$.

Proof: Consider any trihedron *Oabc*. Denote by $[P_a]$ the points of $[P]$ which lie on the half-line a. We show first that there is a finite subset of $[R]$, $R_1, R_2, \cdots, R_{n_1}$, such that every point of $[P_a]$ lies within at least one of the regions $R_1, R_2, \cdots, R_{n_1}$. Since the set of all points common to two closed sets is a closed set it follows that the set $[P_a]$ is closed. If the point ∞ of a is a point of $[P_a]$ it follows that there is a region of the set [R] which contains some infinite segment $K\infty$ of a. It remains to show that all points of $[P_a]$ on OK are contained in some finite set of regions of $[R]$. Beginning at O denote by $[P_a']$ the set of all points of $[P_a]$ on OK which are contained within some finite set of regions. This set of points has a least upper bound $\overline{B}\,[P_a']$, which is a point of

$[P_a]$ since $[P_a]$ is closed. By hypothesis $\overline{B}\ [P_a']$ lies within one region of $[R]$. Then either $\overline{B}[P_a']$ is a limit point of points of $[P_a']$ which are on $\overline{B}[P_a']K$, in which case $\overline{B}[P_a']$ fails to be an upper bound of $[P_a']$, or there is a definite next point of $[P_a]$ on $\overline{B}[P_a']K$ and lying within a region of $[R]$. Hence $B[P_a']$ cannot be an upper bound as just specified. Hence the points of $[P_a]$ on OK which lie within a finite set of regions of the set $[R]$ has no upper bound and therefore there exists a set of regions $R_1, \cdots, R_{n_1}$ of the set $[R]$ such that every point of $[P_a]$ lies within at least one of them.

Obviously there are half-lines k proceeding from O and lying within the angle (a, b) such that every point of $[P]$ which lies on or within the angle (a, b) lies within at least one of the regions $R_1, \cdots, R_{n_1}$. If there does not exist a finite set of regions of the set $[R]$ such that every point of $[P]$ which lies on or within the angle (a, b) lies within at least one of them, then let k' be the bound of all half-lines proceeding from O such that the points of $[P]$ which lie on or within the angle formed by them and a lie within such finite set of regions. By the above proof all points of $[P]$ which lie on k' lie within a finite set of regions of the set $[R]$. Hence k' cannot be a bound as specified, and every point of $[P]$ which lie on or within the angle (a, b) lies within at least one of a certain finite set $R_1, R_2, \cdots, R_{n_2}$ of regions of the set $[R]$. In the same manner we may show that there exists a subset of $[R]$, $R_1, R_2, \cdots, R_{n_3}$ such that every point of $[P]$ which lies on or within the trihedron $Oabc$ lies with in at least one region of the set $R_1, \cdots, R_{n_3}$. But the three planes determined by the faces of the trihedron $Oabc$ separate the three-space into eight trihedrons of the type $Oabc$. whence the theorem follows for the whole set $[P]$.

This theorem has many immediate corollaries, of which I instance the two following.

1. *Every infinite set of points has at least one limit point which may be finite or infinite.*

2. *If $z = f(x, y)$ and if (a, b) is a limit point of the set (x, y) on which $f(x, y)$ is defined then $z = f(x, y)$ has some value approached as (x, y) approach (a, b).*

Proof: If the theorem fails to hold, then about every point of the line $x = a, y = b$ there is some region within which there is no point of the graph of the function $z = f(x, y)$. By the theorem of this note there is a finite subset of these regions and

hence a cylinder enclosing the line $x = a$, $y = b$ within which there is no point of the graph of the function. Hence for values of x and y within this cylinder there are no values of the function, which is contrary to the hypothesis that the function is defined on a set which has (a, b) as a limit point.

§ 3. *A Theorem of Continuity.*

THEOREM: *If* $[P]$ *is a closed bounded set of points and if* $[R]$ *is a set of regions such that every point of* $[P]$ *is an interior point of at least one region of the set then there exists a number* d *such that a cube whose edge is* d *and whose center is any point whatever of the set* $[P]$ *will lie entirely within one of the regions of the set* $[R]$.

Proof: About every point of $[P]$ construct a cube c lying within some region of the set $[R]$. Let the sides of the cubes be parallel to the coördinate planes of a fixed rectangular system. By the Heine-Borel theorem there is a finite subset $c_1, \cdots c_n$ of these cubes such that every point of $[P]$ lies within at least one of them. Obviously, that part of the surface of any cube of $c_1, \cdots, c_n$ which is exterior to all other cubes of the set is made up of a finite number of rectangles. Denote the set of all such rectangles obtained from $c_1, \cdots, c_n$ by $[r]$. No such rectangle contains a point of $[P]$ or is a limit point of $[P]$. Hence there exists a positive number d' which is less than the distance from any point of $[P]$ to any point of $[r]$. Let d' be a positive number less than the distance from any side s of a cube of $c_1, \cdots, c_n$ to any parallel side (the plane of which does not contain s). Let d'' be the smaller of these two numbers then $d = \frac{1}{2}d''$ is the required number.

This theorem appears to have been stated in essentially the above form by several persons during the last year. Professor Bolza used it for a set of points in a plane as early as the spring of 1905, and it is possible that the present statement of it is partly due to his suggestion. Mr. Wedderburn has also used the theorem in this form for linear sets. So far as I know it has not been published before now. Dr. Veblen suggested that this form of the theorem be called "the theorem of uniformity." The following is an immediate corollary: "*A function which is continuous over an interval is uniformly continuous over that interval.*"

CHICAGO,
February 22, 1906.

FUNCTIONS OF REAL VARIABLES.

Leçons sur les Fonctions de Variables réelles et les Développements en Séries de Polynomes. Professées a l'École normale supérieure par ÉMILE BOREL, et rédigées par MAURICE FRÉCHET. Avec des notes par PAUL PAINLEVÉ et HENRI LEBESGUE. Paris, Gauthier-Villars, 1905. viii + 160 pp.

THE theory of functions has for several years been distinctly the fashionable thing in mathematical circles in Paris, and the publication of Borel's collection of monographs on the subject, of which the volume now under review is the eighth, aims to gain devotees elsewhere. For we take the chief purpose of this series of publications to be the encouragement of research by the presentation in connected form of the more recent results in the theory of functions, each volume confining itself to some limited domain of investigation. So the present volume confines itself in the main to a connected exposition of the present state of the theory of series of polynomials of a real variable, connected not merely in itself, but also with the older theory.

This volume, like its predecessors, is characterized by great precision of statement, rigor of proof, and elegance of treatment throughout. If a single word were desired to describe the whole, we could think of no better than "modern." The book is fairly alive with the modern spirit. As an instance we may note the fact that the so-called Heine-Borel theorem, which the reviewer believes should play a fundamental rôle in every modern treatment of the theory of functions of real variables, is introduced at the outset (page 9) and is used extensively throughout the volume, greatly simplifying many proofs and adding largely to the general elegance of the exposition.

As has already been said, one of the characteristics of the volume is precision of statement. It is strange, therefore, that one of the few criticisms that we have to make should be for a lack of precision. On page 6 the author speaks of "the segment (0, 1), which we will call the *fundamental interval*." We are not told however that the end points are to be included. On the following page it becomes evident that 0 and 1 are points of the fundamental interval; and in fact throughout the volume the symbol (a, b) seems to be used for the set of points x satisfying the condition $a \leqq x \leqq b$ (in this review we will so understand the symbol), but this is nowhere stated explicitly, and the

occurrence of the phrase "the interval (a, b), extremities included" (page 27, *e. g.*) seems to imply that the last two qualifying words are necessary. This we believe is a lack of precision in notation. It would be better surely to introduce at the outset unique symbols to denote intervals with end points included, and with one or both excluded.*

As may be inferred from the above, the first of the five chapters that make up the body of the volume concerns itself with the theory of point sets. Many of the definitions and theorems here given have of course been known for some time. And yet ideas of recent date are introduced almost at once and with telling effect in the simplification of older proofs. Thus the theorem of Cantor-Bendixson that any closed set can be resolved into a perfect set and an enumerable set, which was originally proved by the aid of transfinite numbers, receives a simple elementary proof by the use of Lindelöf's notion of a point of condensation.

The notion of the measure of a point set (in the Borel sense) receives extended treatment. It is fundamental in much of what follows. What the author calls the fundamental theorem of measure, viz., that the points of an enumerable set of intervals whose total length is less than unity cannot exhaust all the points of an interval of length one, follows almost immediately from the Heine-Borel theorem, already referred to. The general notion of measure is introduced in practically the same way as in the author's earlier volume, Leçons sur la théorie des fonctions (1898). This method of treatment is in so far unsatisfactory that it does not furnish a general means of determining whether or not a given point set is measurable. However, it is possible to show that every point set which is explicitly defined by certain very general operations is measurable. The author, in fact, goes further. He claims that any set which can be explicitly constructed (que nous pourrons *effectivement* former) must be definable in terms of the operations referred

* Peano has used the symbols $\overline{ab}$, $\overline{ab}$, $\overline{ab}$, $\overline{ab}$, to denote respectively all points x satisfying the conditions $a \leqq x \leqq b$, $a \leqq x < b$, $a < x \leqq b$, $a < x < b$. (Cf. Lezioni di analisi infinitesimale, vol. 1 (1893), p. 9.) More recently Pierpont, Theory of functions of real variables, vol. 1, p. 119, has suggested the symbols (a, b), (a, b^*), (a^*, b), (a^*, b^*) to denote the same sets of points. Moreover, the two words "interval" and "segment" may well be used to distinguish between inclusion and exclusion of end points, the former, say, being used when the end points are included, the latter if one or both of them are excluded. Cf. Veblen, "On the Heine-Borel theorem," BULLETIN, vol. 9 (1904), p. 487.

to, and hence that every set that we can explicitly define is measurable.

Continuity is the subject of the second chapter. The definitions are the usual ones. Some of the fundamental theorems receive more or less obvious generalizations. The theorem, for example, that a function which is continuous in an interval (a, b) is uniformly continuous in the interval, is stated more generally as follows, $\omega(x)$ denoting the oscillation at the point x:

*If in the interval (a, b) we have $\omega(x) \leqq c$, we can associate with every positive number ϵ a positive number η, such that the inequality $|\xi_1 - \xi_2| < \eta$ implies, for any pair of points ξ_1, ξ_2 of (a, b), the inequality $|f(\xi_1) - f(\xi_2)| < c + \epsilon$.**

The proof by means of the Heine-Borel theorem is very simple. The chapter closes with an elegant discussion of Lebesgue's generalization of the notion of integral.

Chapter III. deals with series of real functions, the bulk of the chapter being devoted to a very clear discussion of the conditions under which a convergent series of continuous functions defines a continuous function. The solution of the problem of deriving the necessary and sufficient condition herefor is due to Arzelà. The condition in question, for which Arzelà introduced the term "uniform convergence by segments," is called "quasi-uniform convergence" by M. Borel. Townsend † has used Moore's term "subuniform convergence" as an equivalent, in his recent article on Arzelà's condition; it would seem to be the most convenient. The definition is stated as follows, $r_n(x)$ denoting the remainder after n terms:

A series is said to converge subuniformly in (a, b), if 1) the series converges in (a, b); 2) it is possible to associate with every ϵ, and every N, a finite $N' > N$, such that for every x in (a, b) there exists an integer n_x lying between N and N' such that $|r_{n_x}(x)| < \epsilon$.

Arzelà's theorem states *that the necessary and sufficient condition that a series whose terms are continuous in (a, b) shall define a continuous function in that interval, is that the series shall converge subuniformly in the interval.* This theorem was first given by Arzelà in 1884, but the proof was attacked on the score of rigor. His later proofs (1899 and 1902) which he claims are

*In the original b is used where we have used c, thus introducing ambiguity in the use of b.

† BULLETIN, vol. 12 (1905), p. 17. Cf. also Moore, BULLETIN, vol. 7 (1901), p. 257.

sound are complicated. Under these conditions one must needs admire the extreme simplicity of the proof given by Borel in this volume. It occupies barely two pages.*

The point of departure in the next (fourth) chapter, which deals with the representation of continuous functions by series of polynomials, is naturally the theorem of Weierstrass to the effect that every continuous function can be so represented. Several proofs of this theorem are given, including the original of Weierstrass, that of Volterra, and the purely elementary ones of Runge, Mittag-Leffler and Lebesgue. We find also the extension to functions of several variables, and Borel's theorem on the representation of a continuous function, which has continuous derivatives of every order in the interval $(-1, 1)$ as the sum of a Fourier series and a power series in x. The last eighteen pages of the chapter are devoted to the application of the foregoing theory to the theory of interpolation. A critical examination of Lagrange's formula shows that it can not be relied upon to give an approximation to the given function which increases as the number of known values increases. It is possible, however, *to define once for all polynomials* $P_{p,q}(x)$ *such that any function* $f(x)$, *continuous in* $(0, 1)$, *can be represented by the series, uniformly convergent in* $(0, 1)$,

$$f(x) = \Pi_1 + (\Pi_2 - \Pi_1) + \cdots + (\Pi_n - \Pi_{n-1}) + \cdots,$$

where

$$\Pi_q = \sum_{p=0}^{p=q} f\left(\frac{p}{q}\right) P_{p,q}(x).$$

This is the general formula of interpolation which Borel presented in the summer of 1904 to the Third International Congress of Mathematicians. His procedure is beautifully simple and is briefly as follows:†

Let $\phi_{p,q}(x)$ be a continuous function which in the interval $(0, 1)$ coincides with the function represented by the broken line $OMPNA$ in the figure where P is the point $(p/q, 1)$. It is then clear that if x_1 is any point of the interval $((p-1)/q, p/q)$ and if $\phi_{p,q}(x_1) = \theta$, then $\phi_{p-1,q}(x_1) = 1 - \theta$, and all the other

* We may refer the reader who is interested in the comparison to the account of Arzela's work in the paper by Townsend already mentioned, which also contains several interesting examples illustrative of the theory.

† Borel's treatment is throughout purely analytic. We have given it this geometric phrasing merely for the sake of greater brevity of statement.

$\phi_{i,q}(x_1)$ vanish. Now, if q be taken so large that the inequality $|x_1 - x_2| < 1/q$ insures the inequality $|f(x_1) - f(x_2)| < \epsilon$ throughout the interval (0, 1), then it follows immediately that

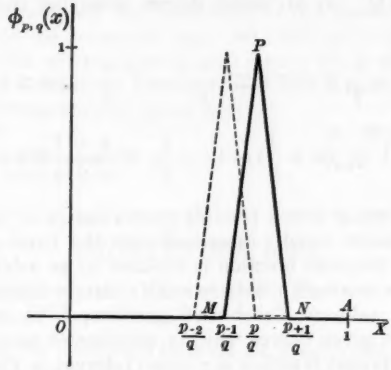
$$\left|\sum_{p=0}^{p=q} f\left(\frac{p}{q}\right)\phi_{p,q}(x) - f(x)\right| < \epsilon$$

throughout (0, 1). It is then only necessary to take for $P_{p,q}(x)$ a polynomial such that the inequality

$$|\phi_{p,q}(x) - P_{p,q}(x)| < 1/q^2$$

holds in (0, 1), to obtain Borel's theorem.

This result is theoretically of much interest. But the reviewer can not agree with M. Borel, when he says in the preface that



this formula is "susceptible of almost immediate application to practical problems of great variety." Any practical application must depend on the explicit calculation of the polynomials $P_{p,q}(x)$, and the nature of the function $\phi_{p,q}(x)$ would seem to require the degree of these polynomials to be very high,* too

* A method for calculating these polynomials has recently been published (M. Potron, "Sur une formule générale d'interpolation," *Bulletin de la Société mathématique de France*, vol. 34 (1906), p. 52) which makes the degree of $P_{p,q}(x)$ equal to $4q^6$. This number may well be capable of reduction by the use of another method for the calculation; but it is hardly open to question that the polynomial of *lowest* degree satisfying the conditions will still be of little value for practical purposes.

high in any case to be of practical value. In this connection the following modification of Borel's procedure may be of interest. His argument will hold word for word if his function $\phi_{p,q}(x)$ be replaced by any continuous function $\psi_{p,q}(x)$, which satisfies the following conditions: (1) it shall vanish for all points of (0, 1), except for points of the segment $[(p-1)/q, (p+1)/q]$, and (2) for all values of x in the interval $[(p-1)/q, p/q]$ it shall satisfy the relation

$$\psi_{p,q}(x) + \psi_{p,q}(x + 1/q) = 1.$$

The function $\phi_{p,q}(x)$ is theoretically the simplest satisfying these conditions, but it is quite possible that it is not the most convenient for the purpose of representation by means of a polynomial. It would be interesting to determine, if possible, the polynomial $Q_{p,q}(x)$ of lowest degree satisfying the following conditions:

$$1)\ |Q_{p,q}(x)| < \frac{1}{q^2} \text{ if } 0 \leqq x \leqq \frac{p-1}{q} \text{ or } \frac{p+1}{q} \leqq x \leqq 1 \ (p<q),$$

$$2)\ |Q_{p,q}(x) + Q_{p,q}(x + \frac{1}{q}) - 1| < \frac{1}{q^2}, \text{ if } \frac{p-1}{q} \leqq x \leqq \frac{p}{q}.$$

This suggests at once a possible generalization of the problem of approximation usually associated with the name of Tchebicheff, where the given function is replaced by an arbitrary function which is required merely to satisfy certain functional relations. The ordinary Tchebicheff problem, *i. e.*, to find that polynomial of given degree which approximates most closely to a given continuous function in a given interval, is given a masterly exposition * in the concluding pages of the chapter.

The eight pages of the fifth and last chapter give a brief introduction to the methods and results recently developed by Baire concerning the representation of discontinuous functions by series of polynomials.

In addition to the five chapters of M. Borel's monograph the volume contains three notes; two short ones by Lebesgue and Borel on "A proof of a theorem by M. Baire" and "On the existence of functions of any class whatever" respectively, and

* On page 87, there is an error of detail; η must be chosen so that $|Q(x)|$ is not only less than $\mu - \varepsilon$, but also less than $\mu - \mu'$.

one of 48 pages by Painlevé on "The development of analytic functions." The latter will perhaps be to some the most interesting part of the book, as in it M. Painlevé gives for the first time a connected development of the results which he has recently published in the *Comptes Rendus* concerning the expansion of analytic functions in series that are valid in the whole complex plane with the exception only of certain half-lines, *i. e.*, in the whole "star of holomorphism." But this review is already too long; we can not go into details.

The following are the only typographical errors we have noticed; page 7, line 2, for F read P, and line 8, for M read N; page 16, line 5 from bottom, for E_1 read E_2, and line 4 from bottom, for E_2 read E_1; page 33, line 9, after "limites" insert "inférieures et"; page 58, line 8 in the formula replace y_i by $(y_i - y_{i-1})$; page 60 in the expansion for $\sqrt{x^2}$ the last factors in the numerators of the coefficients of the second and third terms should be removed; page 80, the equations defining $\phi_{p,q}(x)$ are not correct when $p = 0$; page 82, in the last sentence of the footnote for Kircherberger read Kirchberger; page 85, line 15 from bottom, for A read A'.

J. W. YOUNG.

PRINCETON UNIVERSITY,
March 23, 1906.

SHORTER NOTICES.

Advanced Algebra. By HERBERT E. HAWKES. Ginn and Company, 1905. 285 pp.

IN writing an Advanced Algebra, in the current acceptation of the phrase, two courses are fairly open to an author. He may lay his foundations deep and build on them with unflinching rigor, bringing teacher and pupil into intimate touch with some of the epoch-making researches of the past fifty years. From the standpoint of pure science, this is, of course, admirable; but the meat which it supplies is only for strong men. Foundation-laying is for the hardiest of frontiersmen; and the methodical account of its severely logical steps makes demands upon the average freshman which are far beyond his power to meet.

The second course consists in a frank and explicit assumption of the postulates, as they become necessary in the order of

thought, and a frequent appeal to intuition to bridge over awkward chasms. The words "frank and explicit" are used advisedly, in order to differentiate this second course from a vicious middle one, which has often characterized our text books, namely that of making a show of thoroughness without furnishing the genuine article. Nothing is more fatal to deep and fruitful mathematical thinking than such pseudo-rigor.

The book before us follows very conscientiously the second of the courses noted above. The assumptions, so far from being veiled, appear with refreshing boldness as captions and in italics, and the pages ring true. Incidentally it may be remarked that the clear-cut "euclidean" mode of presentation greatly enhances the value of all the books of this Yale series. Every important discussion has a definite outcome and issues in a clearly stated and italicized proposition.

The author also shows courage in his omissions. He leaves out a number of things which for years have been held to belong to the orthodox canon. We look in vain for cube root, the familiar but forbidding proofs of the generalized binomial expansion, convergency, exponential and logarithmic series and the multiplication theorem for determinants. Derivatives, multiple roots and Sturm's theorem have also disappeared. Many thoughtful teachers will miss some of these things and find the book too meager, but the author would rightly rejoin, in the spirit of his preface, that they require the calculus or other portions of mathematics for their complete comprehension. At any rate it must be admitted that he has made good use of the space saved by these omissions, in devoting it to an unusually extended and interesting treatment of graphical representation. If one is seeking for a fine illustration of the utility of graphs in picturing a highly abstract bit of elementary analysis, he need only turn to the figure on page 105, introduced to illustrate the behavior of the general quadratic when the coefficient of x^2 is infinitesimal.

The book begins with a concise account of the fundamental operations and their laws, then proceeds with a rapid review of the usual topics as far as quadratics. Next follows a chapter on mathematical induction, a method which is immediately applied to the derivation of the binomial theorem in the case of positive integral exponents. From this point on we find the customary material of a manual of the kind, with the omissions already referred to. In nearly every chapter, however, the

subject matter has been refashioned, clarified and enlivened. If, in the modesty of the author's programme, scholarship may not have its perfect work, yet intimations of its existence are everywhere apparent. Above all else he rarely loses the viewpoint of the teacher or forgets the fact that he is forging a classroom tool.

While in this he has been exceptionally successful, his work is not without its faults. There are errors, some minor and others serious and misleading. One of the characteristics of the book is the repeated use of mathematical induction, yet in the first case in which this method is called into service (page 129) it is abandoned at the end for mere inference. A second attempt to use it (page 177) fails badly because of errors in subscripts and the repeated occurrence of the words "negative roots"—an error which is certainly not typographical. The proposition that imaginary roots enter in pairs is laboriously proved, when eight or ten lines would suffice for a satisfactory demonstration. On page 167 the converse of the corollary should have been stated and proved. While the discussion of the generalized binomial theorem has been perhaps wisely omitted, attention should have been called to the fact that it holds for negative and fractional exponents, and examples furnished to illustrate its use in such cases. In the chapter on determinants (page 221) A_1, A_2, etc., are not "minors," as the author has defined this term, but the old-fashioned cofactors, and this error vitiates the proof. The value of the book, in the opinion of the reviewer, would have been increased by the addition of a brief chapter on the "gradients" of curves, immediately after that on graphs. An experience of several years shows that a very few pages would suffice—and few portions of algebra interest students more. An answer book has been compiled but its value is impaired by the errors which it contains.

But such faults as have been indicated can be easily corrected in a subsequent impression and do not invalidate the conclusion that the book under consideration is admirably adapted to the field which it undertakes to cover. Alike in its theory and in its practice it furnishes fresh, rich and wholesome material for the advanced course in fitting schools and first year work in our colleges.

GEORGE D. OLDS.

Opere matematiche di FRANCESCO BRIOSCHI. Publicate per cura del comitato per le onoranze a Francesco Brioschi (G. Ascoli, E. Beltrami, G. Colombo, L. Cremona, G. Negri, G. Schiaparelli). Volumes 1, 2, 3. Milan, U. Hoepli, 1901, 1902, 1904. xi + 416, viii + 456, x + 435 pp.

BRIOSCHI'S life covered seventy-three years (1824–97), the last fifty filled with mathematical labor as teacher, writer, and editor. Taking his doctorate at Pavia in 1845, he became professor of applied mathematics in that university in 1852, and in 1862 was commissioned by the government to found and organize the Istituto tecnico superiore at Milan. There he filled until his death the chairs of hydraulics and analysis, and the office of director. Together with Tortolini he founded in 1858 the *Annali di matematica pura ed applicata* (the immediate successor of the eight volumes of *Annali di scienze matematiche e fisiche*), and carried on independently the enterprise, so significant for the revival of mathematics in Italy, in a second series of twenty-six volumes, 1868–1897. Of his pupils at Pavia the most notable are Cremona, Beltrami, and Casorati. For an account of his labors for the Italian government and a careful analysis of his writings, one should read the excellent biography by Noether in the *Mathematische Annalen*, volume 50 (1898).

The present publication is a part (and the principal part we should esteem it) of the plan to provide a worthy memorial for this great scholar and educator. Scattered in five or more journals, the memoirs embodying Brioschi's extraordinary activity as commentator, critic and creator of modern mathematics had produced their first effect, that on his contemporaries. Now, collected, they become part of the accessible treasure of the scientist and historian. The plan is apparently to collect not his well-known books on determinants and mechanics of rigid systems, but only the shorter papers, from journals and the proceedings of learned societies.

So far there are republished 144 titles. Of these a rough enumeration shows 57 on algebraic invariants, 14 on modern geometry, 14 on differential geometry and the calculus of variations, 19 on differential equations, 21 on the theory of functions and higher analysis, 7 on history, bibliography and biography, 10 on physics or mechanics, and 2 miscellaneous. A more extended résumé is not our purpose here, in view of Noether's paper cited above. This collection reminds one of Cayley's works in

its magnitude, but of Clifford's in its large number of brief, pointed papers on topics of general mathematical interest.

The arrangement is by sets, the articles from any one serial appearing together in chronological order. The monograph on the theory of invariants of binary forms is at the close of volume 1, those on the solution of the quintic and sextic are scattered through all of these volumes. Each has a table of contents and a well-executed index. Volume 1 contains the well-known portrait, and the concluding (fifth) volume is to furnish a classified index and a compendious life.

H. S. WHITE.

Verhandlungen des dritten internationalen Mathematiker-Kongresses in Heidelberg vom 8 bis 13 August, 1904. Herausgegeben von dem Schriftführer des Kongresses, DR. A. KRAZER. Leipzig, Teubner, 1905. Royal 8vo, x + 755 pages.

A CONGRESS that can attract four hundred members and hold them for a week, half of them coming from countries outside Germany, must show in its proceedings something worth the study of philosophers, still more that demands the attention of progressive mathematicians. Numbers count for little, but the names of the participants prove the representative and international character of this great gathering. It is gratifying to see Russia occupying second place in number of delegates, and our own country contesting with Denmark the fifth place.

The secretary's report contains the address of Professor Weber, of Strassburg, in opening the Congress, which, it will be remembered, was designed in part as a commemoration of the centenary of the birth of C. G. J. Jacobi. Then follows the address of Professor Königsberger on Jacobi's life and work — dignified, scholarly, full of particulars, but free from technicalities, in short the finished production of the one man who was qualified by his biographical studies and oratorical ability for this onerous task. Next come the four lectures delivered in the general sessions, and these will doubtless prove, for most readers, the most profitable part of this voluminous record. Their titles are as follows: P. Painlevé, "The modern problem of integration of differential equations"; A. G. Greenhill, "The mathematical theory of the top, considered historically"; C. Segre, "The geometry of the present day and its

connections with analysis"; W. Wirtinger, "Riemann's lectures upon the hypergeometric series and their significance." Of these the first and third belong to that rare and valuable class of mathematical literature which appears in some prefaces and in a few biographies, precise in statement, artistic in diction and style, broadly comprehensive, yet vividly suggestive. The others attain nearly the same degree of interest, Professor Greenhill's by its graphic mingling of experiment with theory, and Professor Wirtinger's by its lucid exposition of rich material preserved in a shorthand report of Riemann's lectures, and now brought to light after more than forty years of oblivion. After reading this and recalling the few similar events now classic, one may venture to expect that some day yet the missing manuscripts of Jacobi, alluded to by Königsberger, may be recovered.

The titles of the papers read before the six sections, nearly eighty in number, were given in the BULLETIN a year ago,* together with many abstracts and summaries. Most of them will appeal to large circles of readers; it may be permissible to mention one in particular that is universally interesting, that of Professor F. Meyer, of Königsberg, on the essential nature of mathematical proof. That there is no real increase of mathematical knowledge has been affirmed by others, but certainly not often reinforced by such a convincing series of far-reaching analyses: to the principal thesis the eloquent closing paragraph may be found to supply the needful antithesis.

H. S. WHITE.

NOTES.

THE April number (volume 7, number 2) of the *Transactions of the* AMERICAN MATHEMATICAL SOCIETY contains the following papers: "On geometries in which circles are the shortest lines," by C. E. STROMQUIST; "A generalization of the notion of angle," by G. A. BLISS; "The square root and relations of order," by O. VEBLEN; "The problem of partial geodesic representation," by E. KASNER; "On the pentadeltoid," by R. P. STEPHENS; "The groups of order p^m which contain exactly p cyclic subgroups of order p^a" by G. A. MILLER; "Groups in which a large number of operators may

* Vol. 11 (1905), pp. 205–217, and 247–263.

correspond to their inverses," by W. A. MANNING; "Finite projective geometries," by O. VEBLEN and W. H. BUSSEY; "On the analytic extension of functions defined by double power series," by W. B. FORD; "On quadratic hermitian and bilinear forms," by L. E. DICKSON; "Die kinematische Erzeugung von Minimalflächen," by P. STÄCKEL; "A fifth necessary condition for a strong extremum of the integral

$$\int_{x_0}^{x_1} F(x, y, y')dx,"$$

by O. BOLZA; "A problem in the calculus of variations in which the integrand is continuous," by G. A. BLISS and M. MASON.

THE April number (volume 7, number 3) of the *Annals of Mathematics* contains: "Introduction to the theory of Fourier's series (concluded)," by M. BÔCHER; "Note on multiply perfect numbers," by R. D. CARMICHAEL; "Note on integrating factors," by E. B. WILSON.

AT the meeting of the London mathematical society held on March 8, the following papers were read: By H. LAMB, "Sommerfeld's diffraction problem and reflection by a parabolic mirror"; by L. J. ROGERS, "Function-sum theorems connected with the series

$$\sum x^n/n^2";$$

by T. J. I'A. BROMWICH, "Investigations on series of zonal harmonics"; by E. W. BARNES, "The functions $g_\beta(x, \theta)$ and $f_\beta(x, \theta)$"; by E. J. NANSON, "Relations between the p-line determinants formable from a p by q array."

THE active participation of the members of the AMERICAN MATHEMATICAL SOCIETY in the organization of associations of teachers of mathematics is partially indicated by the following list of members who are officers of these associations in the middle west or on the Pacific coast: Professor H. C. Harvey, president, Dr. L. D. Ames, secretary, and Professor E. R. Hedrick, chairman of the executive council in the Missouri association; Professor T. E. McKinney, secretary of the Ohio association; Professor W. J. Rusk, chairman of the mathematics section of the Iowa teachers' association; Professor H. E. Slaught, vice-president of the mathematics section of the Central association of science and mathematics teachers; Pro-

fessor G. A. Miller, president of the mathematics section of the California teachers' association; Professor R. E. Moritz, president of the Washington state association of mathematics teachers.

THE seventy-sixth annual meeting of the British association for the advancement of science will be held at York, August 3 to 8. Professor R. LANKESTER is president of the association, and Mr. E. H. GRIFFITHS is president of section A, mathematics and physics.

THE publishing house of Gauthier-Villars in Paris announces the following books in press: J. TANNERY, Leçons d'algebre et d'analyse, volume II; P. JANET, Leçons d'electrotechnique générale, volume III; E. PICARD, Oeuvres de Charles Hermite, volumes II and III.

UNIVERSITY OF CHICAGO.— The following advanced courses in mathematics are announced for the summer quarter, June 19 – September 1. — By Professor O. BOLZA: Elliptic functions, four hours; Functions of a real variable, four hours. — By Professor H. MASCHKE: Projective geometry, four hours. — By Professor H. E. SLAUGHT: Elliptic integrals, four hours; Analytic geometry, five hours. — By Professor L. E. DICKSON: Algebraic analysis, four hours; Theory of substitutions, four hours. — By Dr. A. C. LUNN: Integral calculus, five hours; General Seminar, two hours. — By Mr. N. J. LENNES: Pedagogy of mathematics, four hours.

YALE UNIVERSITY. — The following advanced courses are announced for the year 1906-1907. — By Professor J. PIERPONT: Advanced mechanics, two hours; Advanced theory of functions, two hours; Theory of functions of a real variable, two hours. — By Professor P. F. SMITH: Advanced analytic geometry, two hours; Foundations of geometry, one hour. — By Professor H. E. HAWKES: Linear associative algebra, two hours; Teachers course in geometry, two hours. — By Professor M. MASON: Calculus of variations, two hours; Differential equations of physics, two hours. — By Professor E. B. WILSON: Advanced calculus, two hours; Thermodynamics, two hours.— By Dr. W. A. GRANVILLE: Differential geometry, two hours. — By Dr. L. E. HEWES: Differential equations, one hour; Geometric transformations, two hours. — By Mr. E. L. TAYLOR: Scientific computation, one hour. — By Professor W. B. BEEBE: Celestial mechanics, two hours. — By Professor F. E. BEACH: Vector analysis, one hour (first half year).

THE foreign universities below offer courses in advanced mathematics during the present summer semester as follows:

OXFORD UNIVERSITY (Easter and Trinity terms). — By Professor W. ESSON: Comparison of analytic and synthetic methods in the theory of conics, two hours; Informal instruction in geometry, one hour. — By Professor E. B. ELLIOTT: Theory of functions, I, three hours. — By Professor A. E. H. LOVE: Waves and sound, two hours. — By Mr. A. L. DIXON: Calculus of variations, one hour. — By Mr. H. T. GERRANS: Line geometry, two hours. — By Mr. A. E. JOLLIFFE: Higher analytic geometry, two hours. — By Mr. P. J. KIRKBY: Plane curves, two hours. — By Mr. J. W. RUSSELL: Rigid dynamics, two hours. — By Mr. R. F. MCNEILE: Algebra, two hours. — By Mr. C. E. HASELFOOT: Series and continued fractions, two hours. — By Mr. A. L. PEDDER: Spherical trigonometry, one hour. — By Mr. C. H. SAMPSON; Solid geometry, two hours. — By Mr. C. H. THOMPSON: Differential equations, two hours.

UNIVERSITY OF PARIS. — By Professor É. PICARD: Integrals of differential equations, two hours. — By Professor E. GOURSAT: Ordinary and partial differential equations, two hours. — By Professor L. RAFFY: Applications of partial differential equations to geometry, two hours. — By Professor P. PAINLEVÉ: Analytic mechanics, two hours. — By Professor P. APPELL: Rational mechanics, two hours. — By Professor É. BOREL: Theory of integral functions, one hour. Each professor will hold a weekly conference in mathematics, assisted by Professor J. HADAMARD, Dr. E. BLUTEL and Dr. SERVANT.

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: Analytic functions, four hours; Surfaces and twisted curves, four hours; Exercises in maxima and minima, two hours; Colloquium, two hours; Seminar, two hours. — By Professor G. FROBENIUS: Theory of determinants, four hours; Seminar, two hours. — By Professor F. SCHOTTKY: Differential calculus, four hours; Abelian and theta functions, two hours; Seminar, three hours. — By Professor J. KNOBLAUCH: Applications of elliptic functions, four hours; Analytic geometry, four hours; Line congruences, one hour. — By Professor G. HETTNER: Theory of probabilities, two hours. — By Professor R. LEHMAN-FILHÉS: Analytic mechanics, four hours. — By Professor E. LANDAU: Picard's theorem, four hours. — By Dr. I. SCHUR: Integral calculus, four hours; with exercises, one hour.

UNIVERSITY OF BRESLAU. — By Professor J. ROSANES: Plane analytic geometry, three hours; Seminar, one hour. — By Professor R. STURM: Differential geometry, four hours; Descriptive geometry, two hours; Seminar, one hour. — By Professor A. KNESER: Integral calculus, four hours; with exercises, two hours; Calculus of variations, three hours; Seminar. two hours. — By Professor G. LANDSBERG: Differential calculus, four hours; with exercises, two hours; Elliptic and modular functions, three hours.

UNIVERSITY OF FREIBURG. — By Professor J. LÜROTH: Integral calculus, five hours; Trigonometry, two hours; Seminar, two hours. — By Professor L. STICKELBERGER: Analytic mechanics, five hours; Fourier's series and integrals, two hours; Seminar, two hours. — By Professor A. LOEWY: Theory and application of determinants, four hours; Foundations of geometry, two hours; Exercises in the mathematics of insurance. — By Professor J. WEINGARTEN: Selected chapters from the theory of elastic bodies, two hours. — By Dr. K. SEITH: Descriptive geometry, two hours; with exercises.

UNIVERSITY OF GIESSEN. — By Professor M. PASCH: Algebra, four hours; Invariants, three hours; Seminar, one hour. — By Professor E. NETTO: Plane analytic geometry, four hours; Definite integrals, three hours; Seminar, one hour. — By Professor H. GRASSMANN: Analytic mechanics, I, four hours.

UNIVERSITY OF GÖTTINGEN. — By Professor F. KLEIN: Theory of functions, four hours; Seminar (with Professors Hilbert and Minkowski), two hours. — By Professor D. HILBERT: Differential and integral calculus, four hours; Mechanics of continua, four hours. — By Professor H. MINKOWSKI: Algebra, four hours; Spherical harmonics, two hours. — By Professor C. RUNGE: Differential equations, six hours; Seminar, two hours. — By L. PRANDTL: Graphical methods, two hours. — By Professor M. BRENDEL: Theory of probabilities, four hours; Mathematics of insurance, two hours; Seminar, two hours. — By Professor E. ZERMELO: Partial differential equations of mathematical physics, four hours. — By Dr. M. ABRAHAM: Theory of potential, four hours. — By Dr. G. HERGLOTZ: Analytic geometry, four hours. — By Dr. C. CARATHEODORY: Calculus of variations, four hours.

UNIVERSITY OF GREIFSWALD. — By Professor W. THOMÉ:

Theory of analytic functions, elliptic functions, four hours; Differential geometry, two hours; Seminar, two hours. — By Professor F. ENGEL: Analytic mechanics, I, four hours; Analytic geometry of two and three dimensions, four hours; Differential invariants, one hour; Seminar, two hours.—By Professor K. TH. VAHLEN: Integral calculus, four hours; with exercises, one hour; Determinants, one hour.

UNIVERSITY OF KIEL. — By Professor L. POCHHAMMER: Theory of determinants, four hours; Application of the calculus, four hours; Seminar, one hour. — By Professor L. HEFFTER: Differential and integral calculus, four hours; with exercises, one hour; Algebra, four hours; Applications of elliptic functions, one hour.

UNIVERSITY OF KÖNIGSBERG.—By Professor W. F. MEYER: Plane analytic geometry, three hours; Introduction to higher geometry, four hours; Seminar, two hours. — By Professor A. SCHOENFLIES: Theory of functions, five hours; Seminar, two hours. — By Professor L. SAALSCHÜTZ: Determinants, two hours; Differential calculus, four hours; with exercises, two hours.

UNIVERSITY OF MARBURG. — By Professor K. HENSEL: Theory of numbers, four hours; Differential equations, four hours; Seminar, one hour. — By Professor E. NEUMANN: Elliptic functions, four hours; Fourier's series, two hours; Seminar, two hours. — By Dr. F. v. DALWIGK: Analytic geometry, four hours. — By Dr. H. JUNG: Algebraic solution of equations, four hours.

UNIVERSITY OF MÜNSTER. — By Professor W. KILLING: Analytic geometry, I, four hours; with exercises, two hours; Seminar, one hour. — By Professor R. v. LILIENTHAL: Differential and integral calculus, four hours; Curvature of curves and surfaces, four hours. —By Professor M. DEHN: Mechanics, four hours; Analysis situs, one hour; Seminar, two hours.

UNIVERSITY OF STRASSBURG. — By Professor TH. REYE: Synthetic geometry, two hours; Mechanics, four hours; Seminar, two hours. — By Professor H. WEBER: Calculus and introduction to the theory of functions, four hours; Advanced theory of numbers, three hours; Seminar, two hours. — By Professor M. SIMON: History of mathematics in the middle ages, three hours. — By Professor J. WELLSTEIN: Introduction to

the theory of invariants, two hours; Encyclopedia of elementary mathematics, II, three hours; Seminar, two hours. — By Professor E. TIMERDING: Analytic geometry of space, three hours; Applied mechanics, three hours; Theory of probabilities, one hour. — By Professor S. EPSTEIN: Elliptic functions, two hours; Seminar, two hours.

UNIVERSITY OF TÜBINGEN. — By Professor A. v. BRILL: Mechanics, five hours; Foundations of geometry, three hours; Seminar, two hours. — By Professor H. v. STAHL: Lower analysis, three hours; Theory of functions, three hours; Seminar, two hours. — By Professor L. MAURER: Higher analysis, I, three hours; with exercises, two hours; Linear differential equations, one hour.

UNIVERSITY OF BASEL. — By Professor H. KINKELIN: Applications of the calculus, three hours; Analytic geometry, three hours; Mathematics of insurance, one hour. — By Professor O. SPIESS: Special differential equations, three hours; Determinants, one hour. — By Professor M. GROSSMANN: Descriptive geometry, two hours; Non-euclidean geometry, two hours.

UNIVERSITY OF GENEVA. — By Professor C. CAILLER: Differential and integral calculus, three hours; Rational mechanics, three hours; Seminar, two hours. — By Professor H. FEHR: Projective and descriptive geometry, two hours; Higher algebra, two hours; Seminar, one hour. — By Dr. R. DE SAUSSURE: Geometry of motion, two hours.

PROFESSOR D. HILBERT has been awarded the Cothenius medal of the academy of sciences at Halle.

PROFESSOR E. C. PICKERING, director of the observatory at Harvard University, has been elected a member of the Berlin academy of sciences.

PROFESSOR G. HUBER has been promoted to a full professorship of mathematics at the University of Bern, Switzerland.

PROFESSOR J. H. HAGEN, director of the observatory of the University of Georgetown, has been appointed director of the observatory of the Vatican.

DR. W. SCHLINK, of the technical school at Darmstadt, has been appointed professor of mechanics at the technical school at Braunschweig.

UPON the invitation of King Oscar, Professor V. VOLTERRA, of the University of Rome, gave a series of lectures on the differential equations of mathematical physics at the University of Stockholm during the month of February.

DR. VANECEK has been appointed docent in mathematics at the Bohemian technical school at Prague.

DR. G. Z. GIAMBELLI has been appointed docent in projective geometry at the University of Genoa.

DR. G. DE FRANCHIS has been appointed to an associate professorship of algebra and analytic geometry at the University of Cagliari.

PROFESSOR G. FUBINI, of the University of Catania, has been transferred to the University of Genoa, as associate professor of the calculus.

PROFESSOR OSKAR BOLZA, of the University of Chicago, who is spending the year in travel and study in southern Europe and in Egypt, will resume his lectures at the university at the beginning of the summer quarter, June 19, 1906.

MR. L. A. MARTIN, JR., has been promoted to an assistant professorship of mathematics and mechanics at the Stevens Institute of Technology.

PROFESSOR E. L. RICHARDS, of Yale University, will retire from active service at the close of the present academic year.

PROFESSOR J. M. PEIRCE, Perkins professor of mathematics and astronomy and senior member of the faculty of Harvard University, died March 21, at the age of 71 years. He was appointed tutor in mathematics in 1854, university professor of mathematics in 1869, and Perkins professor of mathematics and astronomy in 1885. Professor Peirce had been a member of the AMERICAN MATHEMATICAL SOCIETY since 1898.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMODEO (F.). Vita matematica napoletana. Studio storico, biografico, bibliografico. Parte prima. Napoli, Giannini, 1905. 8vo. 216 pp. L. 10.00

BACHET (C. G.). Problèmes plaisants et délectables qui se font par les nombres. 4e édition, revue et simplifiée. Paris, Gauthier-Villars, 1905. 16mo. 6 + 161 pp. Fr. 3.50

BAUER (G.). Von der Kurve sechster Ordnung, welche der Ort der Brennpunkte der Kegelschnitte ist, welche durch vier Punkte gehen. München, 1905. 8vo. 11 pp.

BERRY (W. J.). Differential equations of the first species. New York, Van Nostrand, 1906. 12mo. Cloth.

BIANCHI (L.). Teoria delle trasformazioni delle superficie applicabili sulle quadriche rotonde. Roma, 1905. 4to. 74 pp.

BÜCHEL (C.). Ganzzahlige Werte bei Diophant. Den Teilnehmern der 48sten Versammlung der Philologen und Schulmänner zu Hamburg 1905 als Festgabe dargeboten von dem Ortskomitee. Hamburg, Herold, 1905. 8vo. 16 pp. M. 1.00

COUTURAT (L.). Les principes des mathématiques, avec un appendice sur la "Philosophie des mathématiques" de Kant. Paris, Alcan, 1905. 8vo. 8 + 311 pp. Fr. 5.00

CZUBER (E.). Vorlesungen über Differential- und Integralrechnung. 1er Band, 1te Hälfte. 2te, sorgfältig durchgesehene Auflage. Leipzig, Teubner, 1906. 8vo. 256 pp. M. 6.00

FASSBINDER (C.). Théorie et pratique des approximations numériques. Paris, Gauthier-Villars, 1906. 8vo. 6 + 92 pp. Fr. 3.00

GOMES TEIXEIRA (F.). Tratado de las curvas especiales notables. Memoria premiada por la Real Academia de Ciencias Exactes, Fisicas y Naturales de Madrid, y publicada por la misma Academia. Madrid, 1905. 9 + 632 pp.

HALDANE (E. S.). Descartes, his life and times. New York, Dutton, 1905. 8vo. 28 + 398 pp. Cloth. $4.50

HAMBURGER (A.). Ueber die Restabschätzung bei asymptotischen Darstellungen der Integrale linearer Differentialgleichungen zweiter Ordnung. (Diss.) Berlin, 1905. 8vo. 58 pp.

HILL (G. W.). Collected mathematical works. (In four volumes.) Vol. 2. Washington, Carnegie Institution, 1904. 4to. 5 + 339 pp. Cloth. $2.50

JACOBSTHAL (E.). Anwendungen einer Formel aus der Theorie der quadratischen Reste. (Diss.) Berlin, 1906. 8vo. 39 pp.

MACAULAY (F. S.). Geometrical conics. 2d edition. Cambridge, University Press, 1906. 8vo. 310 pp. Cloth. 4s. 6d.

MUIR (J.). Theory of determinants; in the historical order of development. Part 1: General determinants up to 1841. Part 2: Special determinants up to 1841. 2d edition, revised. London, Macmillan, 1906. 8vo. 504 pp. Cloth. 17s.

OSGOOD (W. F.). Lehrbuch der Funktionentheorie. (In 2 Bänden.) 1er Band, 1te Hälfte. Leipzig, Teubner, 1906. 8vo. 306 pp. M. 7.00

SAUER (R.). Eine polynomische Verallgemeinerung des Fermatschen Satzes. (Diss.) Giessen, 1905. 8vo. 18 pp.

SIMON (M.). Methodik der elementaren Arithmetik in Verbindung mit algebraischer Analysis. Leipzig, Teubner, 1906. 8vo. 4 + 108 pp.

TESAŘ (L.). Elemente der Differential- und Integralrechnung. Hilfsbuch für mathematischen Unterricht zum Gebrauche an höheren Lehranstalten. Leipzig, Teubner, 1906. 8vo. 8 + 128 pp.

THALREITER (F.). Auflösung gewisser algebraischer Eliminationsaufgaben durch Benützung der Teilungsgleichungen der p-Funktion. (Diss.) München, 1905. 8vo. 59 pp.

THOMAE (J.). Grundriss einer analytischen Geometrie der Ebene. Leipzig, Teubner, 1906. 8vo. 10 + 184 pp.

TOEPLITZ (O.). Ueber Systeme von Formen, deren Funktionaldeterminante identisch verschwindet. (Diss.) Breslau, 1905. 8vo. 46 pp.

VALEWINK (G. C. A.). Over asymptotische ontwikkelingen. Utrecht, 1905. 8vo. 138 pp. M. 3.00

VREESWIJK (J. A.). Involuties op rationale krommen. Utrecht, 1905. 8vo. 104 pp. M. 2.50

VOGT (H.). Eléments de mathématiques supérieures, à l'usage des physiciens, chimistes et ingénieurs, et des élèves des facultés des sciences. 3e édition. Paris, Vuibert, 1906. 8vo. 7 + 620 pp. Fr. 10.00

WALLNER (C. R.). Die Verteilung der Primzahlen nach neuen Gesichtspunkten behandelt. (Diss.) München, 1905. 8vo. 53 pp.

WESTFALL (W. D. A.). Zur Theorie der Integralgleichungen. (Diss.) Göttingen, 1905. 8vo. 67 pp.

WINKLER (F.). Die infinitesimalen Transformationen, welche einen Pfaffschen Ausdruck absolut oder modulo eines vollständigen Differentials invariant lassen. (Diss.) Leipzig, 1905. 8vo. 38 pp. M. 1.50

WÜNSCHMANN (K.). Ueber Berührungsbedingungen bei Integralkurven von Differentialgleichungen. (Diss., Greifswald.) Leipzig, Teubner, 1905. 8vo. 36 pp.

II. ELEMENTARY MATHEMATICS.

BAUMBERGER (G.). Lehrbuch der Planimetrie mit Aufgabensammlung für Mittelschulen, insbesondere technische Lehranstalten. Bern, Stämpfli, 1906. 8vo. 6 + 82 pp. M. 1.60

BOZAL OVEJERO (A.). Tratado elemental de geometría. Madrid, Hernando, 1905. 8vo. 296 pp. Fr. 10.00

COB (A.). See VÁSQUEZ FIGUEROA (A.).

DALMAN Y CARLES (J.). Aritmética razonada y nociones de algebra. Tratado teórico-práctico demostrado con aplicación á las diferentes cuestiones mercantiles para uso de las escuelas normales y de las de comercio. Comprende, además de la teoría indispensable, más de 5000 ejercicios y problemas aritméticos, algebraicos y geométricos para el cálculo mental y escrito. 11a edición corregida. Libro del alumno. Grado profesional. Gerona, Dalman, Carles, y Ca, [1906]. 8vo. 515 pp. Fr. 6.50

DEAKIN (R.). Algebra, preliminary certificate edition, with a supplement on graphs. London, Clive, 1906. vo. 482 pp. Cloth. 3s.

ELSÄSSER (W.). Leitfaden der Stereometrie. Ein Hilfsbuch zum Gebrauch beim Unterricht an höheren Lehranstalten. Stuttgart, Grub, 1906. 8vo. 8 + 90 pp. M. 1.50

GAJDECZKA (J.). Uebungsbuch zur Geometrie in den oberen Klassen der Mittelschulen. 3te, inhaltlich unveränderte Auflage. Wien, Deuticke, 1906. 8vo. 4 + 188 pp. M. 2.20

HEPTNER (L.). Tafeln für Wertsberechnungen; Multiplicationstafeln. Leobschütz, 1905. 8vo. 8 + 201 pp. Cloth. M. 5.80

JIMÉNEZ DE GONZÁLEZ (A.). Nociones de aritmética y geometría. Obra de texto acomodada al programa adoptado para la enseñanza de esta asignatura en academias, colegios é instituciones docentes. Primer curso. Madrid, Fé, 1905. 8vo. 46 pp. Fr. 4.00

KEMLEIN (G.). Zum Unterricht in der analytischen Geometrie an den humanistischen Gymnasien Bayerns. (Progr.) Ludwigshafen, 1905.

LAISANT (C. A.). Initiation mathématique, ouvrage étranger à tout programme, dédié aux amis de l'enfance. Paris, Hachette, 1906. 12mo. 7 + 167 pp. Fr. 2.00

LONDON UNIVERSITY intermediate mathematics papers, 1895–1905, and miscellaneous questions. London, Clive, 1906. 8vo. 114 pp. Cloth. 2s.

—— intermediate mathematics examination papers, 1895–1905. London, Clive, 1906. 8vo. 144 pp. Cloth. 2s.

——. Matriculation model answers: mathematics. Papers from Sept. 1902 to Jan. 1906. London, Clive, 1906. 8vo. 146 pp. 2s.

MARTINI ZUCCAGNI (A.). Trattato d'algebra, con i complementi di aritmetica razionale, ad uso dei licei. 2a edizione. Livorno, Giusti, 1905. 8vo. 16 + 381 pp. L. 2.60

MÜLLER (H.). Die Elemente der ebenen Trigonometrie, mit einer Sammlung von Aufgaben und deren Lösungen. 4te Auflage. Metz, Scriba, 1905. 8vo. 6 + 42 pp. M. 1.20

PLATH (J.). Sammlung von Aufgaben zur Vorbereitung auf die Mittelschullehrer-Prüfung und auf das Arbiturientenexamen am Realgymnasium. Leipzig, Teubner, 1906. 8vo. 8 + 259 pp. Cloth. M. 4.00

RUIZ TAPIADOR (A.). Elementos de algebra. 1a edición. Toledo, Serrano, 1905. 8vo. 9 + 207 pp. Fr. 8.00

SCHLESSER (E.). Trigonométrie rectiligne. Classes de première et de mathématiques. Préparation à l'Ecole navale, à l'Ecole spéciale militaire de Saint-Cyr, etc. Programmes du 27 juillet 1905. Paris, Delagrave, 1906. 8vo. 6 + 353 pp. Fr. 4.00

SPIEKER (J.). Lehrbuch der ebenen Geometrie mit Uebungs-Aufgaben für höhere Lehranstalten. Ausgabe C. Abgekürzte Kurse. 3te Auflage. Potsdam, Stein, 1906. 8vo. 4 + 206 pp. M. 2.50

VÁZQUEZ FIGUEROA (A.) y COB (A.). Opúsculo de problemas de geometría con un apéndice con la rectificación de la circunferencia, trazado y rectificación de la elipse, óvalo, ovoide y espiral con arreglo al último programa de ingreso en el cuerpo de telégrafos. Madrid, 1905. 8vo. 88 pp Fr. 3.25

III. APPLIED MATHEMATICS

ABBE (E.). Gesammelte Abhandlungen. 2ter Band: Wissenschaftliche Abhandlungen aus verschiedenen Gebieten, Patentschriften, Gedächtnisreden. Jena, 1906. 8vo. 4 + 346 pp. M. 7.50

APPELL (P.) et CHAPPUIS (J.). Leçons de mécanique élémentaires, conformes aux programmes du 31 mai 1902. 2 volumes se vendant séparément. Paris, Gauthier-Villars, 1905. Vol. 1, Fr. 2.75
Vol. 2, Fr. 4.00

ASCHKINASS (E.). See FLEMING (J. A.).

BASTINE (P.). See LAUENSTEIN (R.).

BERGEMANN (R.). Reaktionen auf Schalleindrücke nach der Methode der Häufigkeitskurven bearbeitet. (Diss.) Leipzig, 1905. 8vo. 42 pp.

CHAPPUIS (J.). See APPELL (P.).

DALBY (E.). Valve and valve gear mechanisms. London, Arnold, 1906. 8vo. 12 + 366 pp. Cloth. 25s.

DARIÈS (G.). Mécanique, hydraulique, thermodynamique. 2e édition, entièrement remaniée et très augmentée. Paris, Dunod, 1906. 16mo. 8 + 804 pp. Fr. 15.00

DAWSON (M. M.). Practical lessons in actuarial science. Elementary textbook. 2d revised edition. Two volumes. New York, 1905. 8vo. Cloth. $8.50

DREYER (J. L. E.). History of the planetary systems from Thales to Kepler. Cambridge, University Press, 1906. 8vo. 454 pp. Cloth. 10s. 6d.

DRUIFF (G. E.). Refraction. A simple and concise treatise. 2d edition. London, Kimpton, 1906. 8vo. 10s. 6d.

FIEDLER (K.). Gekoppelte Systeme. Rostock, 1905. 8vo. 42 pp. M. 1.50

FISCHER (E.). Ueber künstliche Belastungen bei der Aufstellung von Bogenbrücken. (Diss.) Dresden, 1905. 8vo. 47 pp.

FISCHER (O.). Ueber die Bewegungsgleichungen räumlicher Gelenksysteme. Leipzig, 1905. 8vo. 88 pp.

FLEMING (J. A.). Elektrische Wellentelegraphie. Vier Vorlesungen. Autorisierte deutsche Ausgabe von E. Aschkinass. Leipzig, Teubner, 1906. 8vo. 4 + 185 pp.

FÖRSTER (E.). Vergleichende Untersuchungen an Kreiselpumpen. (Diss.) Dresden, 1905. 8vo. 57 pp.

GAUSS (F. G.). Die trigonometrischen und polygonometrischen Rechnungen in der Feldmesskunst. 3te Auflage. (9 Hefte.) Heft 1. Halle, 1906. 8vo. M. 3.50

GERLAND (E.). See LEIBNIZ (G. W.).

HELMERT (F. R.). Ueber die Genauigkeit der Kriterien des Zufalls bei Beobachtungsreihen. Berlin, 1905. 8vo. 19 pp.

KAUFMANN (W.). Ueber die Konstitution des Elektrons. Berlin, 1905. 8vo. 8 pp.

LAMB (H.). Hydrodynamics. 3d edition. Cambridge, University Press, 1905. 8vo. 648 pp. Cloth. 26s.

LAUENSTEIN (R.). Die graphische Statik. Elementares Lehrbuch, für den Schul- und Selbstunterricht, sowie zum Gebrauch in der Praxis. 9te Auflage. Bearbeitet von P. Bastine. Stuttgart, Kröner, 1906. 8vo. 8 + 253 pp. Cloth. M. 6.00

LEIBNIZ (G. W.). Leibnizens nachgelassene Schriften physikalischen, mechanischen und technischen Inhalts, herausgegeben und mit erläuternden Anwendungen versehen von E. Gerland. Leipzig, Teubner, 1906. 8vo. 6 + 256 pp. (Abhandlungen zur Geschichte der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Heft 21.) M. 10.00

LINKER (A.). Elektrotechnische Messkunde. Berlin, Springer, 1906. 8vo. 8 + 442 pp. M. 10.00

LINNEMANN (M.). Ueber nichtsphärische Objektive. (Diss.) Göttingen, 1905.

LOVE (A. E. H.). Treatise on the mathematical theory of elasticity. 2d edition. Cambridge, University Press, 1906. 8vo. 570 pp. Cloth. 18s.

MANNA (M.). Attrito e rigidezza delle funi e catene. Bari, Laterza, 1905. 8vo. 18 pp.

METZ (G. DE). La double réfraction accidentelle dans les liquides. Paris, Gauthier-Villars, 1906. 8vo. 100 pp. Fr. 2.00

MEUTH (H.). Kinetik und Kinetostatik des Schubkurbelgetriebes. (Diss.) Karlsruhe, 1905. 8vo. 75 pp.

POCKELS (E.). Lehrbuch der Kristalloptik. Leipzig, Teubner, 1906. 8vo 10 + 520 pp.

REINER (J.). Hermann von Helmholtz. Leipzig, 1905. 8vo. 204 pp. M. 3.50

RUIZ CASTIZO (J.). Programa de las lecciones de mecánica racional, explicadas en la Facultad de ciencias. Madrid, Suárez, 1905. 8vo. 28 pp. Fr. 1.00

SCHNITZLER (H.). Ueber die Belichtung von krummen Flächen, speciell von Rotationsflächen zweiter Ordnung. Rostock, 1905. 8vo. 37 pp. M. 1.50

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